MARYLAND

COLLEGE PARK CAMPUS

FINITE ELEMENT SOLUTION TO THE HELMHOLTZ EQUATION WITH HIGH WAVE NUMBER

PART II: THE h-p VERSION OF THE FEM

by

Frank Ihlenburg

and

Ivo M. Babuška

DTIC ELECTE FEB 02 1995 S G D

Technical Note BN-1173

19950131 075

June 1994

INSTITUTE FOR PHYSICAL SCIENCE AND TECHNOLOGY

DISTRIBUTION STATEMENT A

Approved for public release; Distribution Unlimited

Encl (1)

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Note BN-1173	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Finite Element Solution to the Helmholtz Equation with High Wave Number Part II: The h-p Version of the FEM		5. TYPE OF REPORT & PERIOD COVERED Final Life of Contract
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Frank Ihlenburg[1] and Ivo M. Babuska[2]		8. CONTRACT OR GRANT NUMBER(s) [2] N00014-93-I-0131 ONR & #517 402 524 3/DAAD
9. PERFORMING ORGANIZATION NAME AND ADDRESS Institute for Physical Science and Technology University of Maryland College Park, MD 20742-2431		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy Office of Naval Research Arlington, VA 22217		12. REPORT DATE June 1994
		13. NUMBER OF PAGES 60
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release: distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)		

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper, which is part II in a series of two, the investigation of the Galerkin finite element solution to the Helmholtz equation is continued. While part I contained results on the h-verson with piecewise linear approximation, the present part deals with approximation spaces of order $p \geq 1$. The method is assumed to be uniform both w.r. to h and p. The results are presented on a one-dimensional model problem with Dirichlet/Robin boundary conditions. In particular there are proven stability estimates, both w.r. to data of higher regularity and data that is bounded in lower norms.

Finite Element Solution to the Helmholtz Equation with High Wave Number
Part II: The h-p-version of the FEM

Frank Ihlenburg Ivo Babuška

Institute for Physical Science and Technology,
University of Maryland at College Park, College Park MD 20742

For
RA&I
AB
ıced
on
ɔn /
...lability Codes
Dist | Avail and / or Special
A-1

ABSTRACT

In this paper, which is part II in a series of two, the investigation of the Galerkin finite element solution to the Helmholtz equation is continued. While part I contained results on the h-version with piecewise linear approximation, the present part deals with approximation spaces of order $p \geq 1$. The method is assumed to be uniform both w.r. to h and p. As in part I, the results are presented on a one-dimensional model problem with Dirichlet/ Robin boundary conditions. In particular there are proven stability estimates, both w.r. to data of higher regularity and data that is bounded in lower norms. The estimates are shown both for the continuous and the discrete spaces under consideration. Further there is proven a result on the phase difference between the exact and the Galerkin finite element solutions for arbitrary p that had been previously conjected from numerical experiments. These results and further preparatory statements are then employed to show error estimates for the Galerkin finite element method. It becomes evident that the error estimate for higher approximation can - with certain assumptions on the data - be written in the same form as for the piecewise linear case, namley, as the sum of the error of best approximation plus a pollution term that is of the order of the phase difference. The paper is concluded by a numerical evaluation.

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

1 Introduction

This paper is the second part (of two) of an investigtion devoted to the numerical analysis of the Galerkin finite element method for the reduced wave (Helmholtz) equation. The interest in this topic has grown over the last years; related results have been published by a number of authors, both in the mathematical and in the engineering literature [AKS], [De] [DSSS], [Bu], [TP1], [TP2], [HH]. In part I of the present investigation [IB], we analyzed the h-version of the Galerkin finite element method with piecewise linear approximation (i.e. elements of polynomial degree $p = 1$) on the following one-dimensional model problem:

Let $\Omega = (0,1)$ and consider the boundary value problem;

$$\begin{aligned} u''(x) + k^2 u(x) &= -f(x) && (1.1)\\ u(0) &= 0 && (1.2)\\ u'(1) - iku(1) &= 0. && (1.3) \end{aligned}$$

or, equivalently, the variational problem: Find $u \in H^1(\Omega)$, $u(0) = 0$ such that

$$B(u,v) = \int_0^1 \left(u'(x)\bar{v}'(x) - k^2 u(x)\bar{v}(x)\right) dx - iku(1)\bar{v}(1) = \int_0^1 f(x)\bar{v}(x)dx \qquad (1.4)$$

holds for all $v \in H^1(\Omega)$, $v(0) = 0$.

Since we will analyze specifically the case of large wavenumber k, we assume throughout the paper that $k \geq 1$.

We showed [IB]:

A1. For data $f \in L^2(0,1)$, the BVP (1.1-1.3) has unique solution $u \in H^2(0,1)$.

A2. For data $f \in H^1(0,1)'$, the VP (1.4) has unique solution $u \in H^1(0,1)$.

A3. The form B defined by (1.4) is continuous for each fixed wavenumber k:

$$|B(u,v)| \leq C_o(k)|u|_1|v|_1$$

with $C_o = 1 + k + k^2$.

A4. The Babuška-Brezzi constant γ of the VP (1.4) is inversely proportional to the wavenumber k.

Remark 1: Obviously, the statements A1 and A2 hold true for higher regularity of the data; in general, the operators given by BVP (1.1-1.3) and VP (1.4) define bijective

mappings from H^s to H^{s+2}.

For the Galerkin finite element method with piecewise linear approximation ($p = 1$) we then proved:

F1. The finite element solution is (asymptotically) quasioptimal w.r. to the H^1-seminorm provided k^2h is sufficiently small, where h is the size of the elements.

F2. The discrete B-B-condition holds with $\gamma_h \sim \frac{1}{k}$.

F3. In the preasymptotic range, the error bound in H^1-seminorm consists of a term of order kh (reflecting the error of best approximation) and a pollution term of order k^3h^2 (reflecting the phase difference between exact and finite element solutions).

Finally, we conluded from a numerical evaluation that the terms mentioned in F3 do indeed occur in the error and, consequently, the estimate (obtained from F3. by setting $\theta = kh$)

$$|e|_1 \leq C_1\theta + C_2k\theta^2, \tag{1.5}$$

where $C_1\theta$ is the minimal error in the approximation space, cannot be principally improved.

In the present paper we extend our study to elements of arbitrary (but fixed) polynomial order p. Due to the oscillatory character of the propagating solutions, the application of higher order (egs., quadratic or cubic) elements is considered a natural choice in many applied computations (cf. [TP1], [TP2] and references therein). However, except for an asymptotic error estimate, no results stemming from the numerical analysis of Galerkin-type FEM for Helmholtz problems seem to be available for $p > 1$. In the abovementioned asymptotic estimate, which is given without rigorous proof in [BayGTu], it is assumed that $k^2h \ll 1$ (cf. Theorem 3.2 of this paper). As we showed in part I, this assumption on the meshsize does not cover the size of practically applied meshes where the wavenumber k (but not k^2, which is a significant difference for large k) is commonly normalized[1] by the stepwidth h [TP1], [HH].

Consequently, as in part I, it is the principal goal of the present investigation to show error estimates that hold under assumptions on hk only. We call these estimates "preasymptotic" in order to distinguish the results from the abovementioned asymptotic estimates. In section 3 of this paper, we prove two error estimates that

[1] i.e. h is choosen such that the product hk remains close to some apriori fixed value

hold in the preasymptotic range. Both estimates are written as a sum of the error of the best approximation (naturally, the estimates coincide in this first member) and a pollution term. While in the first statement the pollution term is obtained by using an L^2-estimate of the data, the second estimate employs a dual argument to raise the degree of hp^{-1} in the pollution term. Under certain additional assumptions, namely, if the exact solution is sufficiently regular and oscillating with frequency k, we see that the pollution term of the second estimate has (in k, h and p) the order of the phase difference between the exact and the finite element solutions.

It is this phase difference that has been used as the principal criterium for the quality of various finite element approaches [HH], [TP1], [TP2]. We show here (Theorem 3.2) that, for arbitary p, the phase difference between the exact and the Galerkin finite element solutions is of order $k\mathcal{O}\left(\frac{kh}{2p}\right)^{2p}$. To our knowledge, this statement has not been previously proven. It is in accordance with published computational results that have been obtained by Taylor series expansion in the finite element matrices for several fixed p [TP1].

Throughout the paper, special care is given to state, as precisely as possible, the dependence of the constants involved in the estimates on the principal parameters k, h and p.

It turns out (both theoretically and in the numerical evaluation) that the constant in the best approximation estimate grows only moderately with p. On the other hand, the constant in the estimate of the pollution in dual norms (Theorem 3.7) theoretically grows significantly with increasing p. However, this growth has not been observed in several numerical experiments; it is therefore an open question whether this estimate is sharp w.r. to p.

The paper is organized as follows. In section 2 we fix notations and discuss a definition of negative norms that is suited for our purposes. Further, we discuss auxiliary problems and conclude the section with the proof of stability results. In section 3 we investigate the Galerkin finite element solution to the model problem. First (subsection 3.1) we identify the approximation spaces $S_h^p(\Omega)$ and give an outline of the finite element solution procedure (subsection 3.2). In subsection 3.3 we prove the proposition on the phase lag (Theorem 3.2). Subsection 3.4 is devoted to stability propositions on the approximation subspaces. We first have a result that is true for general L^2-data f (Theorem 3.3); in the proof we employ an inf-sup-condition for arbitrary order of approximation (Lemma 3.4). We then show a dual theorem for specific 'bubble' data. This is motivated as follows. In the approximation theorem (Theorem 3.1) we had constructed, for a given function u, an approximizing function s that is of optimal order of approximation in in H^1-, L^2-, and dual norms and, beyond that, posesses specific

interpolation properties w.r. to u. It is these interpolation properties that we make use of in the error analysis where we show that the finite element error is majorized by certain norms of the difference $u - s$. Since $u - s$ occurs as the data in variational problems related to the error estimation, we have to prepare the error analysis with stability specific propositions. Thus prepared, we turn to the error analysis of the Galerkin finite element method (subsection 3.5). We show an asymptotic estimate and then, in light of the previous discussion, turn to a study of the finite element solution in the preasymptotic range.We show that, for oscillating solutions, the pollution term is of the order of the phase lag.

Finally, in section 4, we discuss results of selected numerical experiments that have been carried out to evaluate the error estimates of section 3.

2 Analytic solution properties

2.1 Notations and Preliminaries

We will use the following *notations*:

- **Constants:** If not stated otherwise, all constants $C, D, E, \ldots$ or C_i, where i is a natural number, are understood to be generic, independent of all parameters of the given estimate, and having, in general, different meanings in different context. However, some specific constants that are used repeatedly with the same meaning in different context are marked by literal subscript; so, we write C_a for a constant arising from the principal approximation estimate in the discrete space, and so forth.

- $L^2(0,1)$ is the space of all square-integrable complex-valued functions equipped with the inner product
$$(v, w) := \int_0^1 v(x)\bar{w}(x)dx$$
and the norm
$$\|w\| := \sqrt{(w, w)}.$$

- $H^s(0,1)$ denotes the Sobolev space
$$H^s = \left\{u \,\middle|\, u \in L^2 \wedge \partial^i u \in L^2, i = 1 \ldots s\right\}$$
where $\partial^i u$ are the derivatives of order i in the distribution sense. As usual, we define the subspace
$$H_o^s(0,1) = \left\{u \in H^s(0,1) | \partial^i u(0) = \partial^i u(1) = 0, i = 0 \ldots s-1\right\}.$$

We will also work with subspaces $H^s_{(o}$ and $H^s_{o)}$ consisting of functions with Dirichlet data 0 given only in $x = 0$ or $x = 1$, resp. By $|u|_s := \|\partial^s u\|$ a seminorm is given on H^s. A norm of the space $H^s(0,1)$ is defined as $\|u\|_s := (\sum_{i=0}^s |u|_s^2)^{1/2}$. On H^s_o, $H^s_{(o}$ and $H^s_{o)}$ the seminorm $|\cdot|_s$ is a norm equivalent to $\|\cdot\|_s$.

- $H^{-s}(0,1) = (H^s(0,1))'$ denotes the dual to $H^s_{(o}$ space equipped with the norm

$$\|f\|_{-s} := \sup_{v \in H^s_{(o}} \frac{(f,v)}{|v|_s}. \tag{2.1}$$

- As usual, $f^{(i)}$, $i = 0,1,2,\ldots$, denotes the i-th derivative of f. We generalize this notation for integration: $f^{(-i)}$ is a function s.t. $\partial^i f^{(-i)}(x) = f(x)$. More specifically, we define for $f \in L^2(\Omega)$ and $i = 0,1,\ldots$

$$f^{(-i-1)}(x) = -\int_x^1 f^{(-i)}(t)dt.$$

With these definitions, the dual H^l-norm of a L^2-function f is equal to the L^2-norm of $f^{(-l)}$:

Lemma 2.1 *For $f \in L^2(\Omega)$ and $l = 0,1,2,\ldots$*

$$\|f\|_{-l} = \|f^{(-l)}\| \tag{2.2}$$

holds.

Proof : Let $l = 1$ and write $F = f^{(-1)}$. Then, by partial integration,

$$\|F\| = \sup_{v \in L^2(\Omega)} \frac{\int_\Omega Fv}{\|v\|} = \sup_{v \in L^2(\Omega)} \frac{\int_\Omega fV}{\|v\|}$$

holds with $V := \int_0^x v(t)dt$. Obviously $V \in H^1_{(o}(\Omega)$ and $V' = v$. On the other hand, every $V \in H^1_{(o}(\Omega)$ can be represented by an integral of a L^2-function, hence

$$\|F\| = \sup_{v \in H^1_{(o}(\Omega)} \frac{\int_\Omega Fv}{\|v'\|} = \|f\|_{-1}.$$

This proves the statement for $l = 1$. The induction to higher l is obvious and the proof is completed. $\lhd$

Variational forms arising from the Helmholtz equation are, in general, indefinite. One can obtain, however, coercive forms if the wavenumber is properly restricted (we write K in order to distinguish this problem from the general case where k may be large). For later use, we consider here the case of Dirichlet boundary conditions:

Find $u \in V = H_o^1(\Omega)$ such that for all $v \in V$

$$B_K(u,v) = \int_0^1 u'\bar{v}' - K^2 \int_0^1 u\bar{v} = (f,v). \tag{2.3}$$

holds.

Lemma 2.2 *Let $u \in V = H_o^1(\Omega)$ be the solution to the VP (2.3) with data f. Assume that $0 \le K \le \alpha < \pi$, then*

$$\|u\| \le \frac{1}{\pi^2 - \alpha^2} \|f\| \tag{2.4}$$

$$\|u'\| \le \frac{\pi}{\pi^2 - \alpha^2} \|f\| \tag{2.5}$$

$$\|u'\| \le \frac{\pi^2}{\pi^2 - \alpha^2} \|f^{(-1)}\|. \tag{2.6}$$

Proof : All inequalities are trivial for $u \equiv 0$, hence we may, without loss of generality, assume $\|u\| > 0$. It is easy to see that, for all $u \in H_o^1(\Omega)$,

$$\|u'\| \ge \pi \|u\|.$$

Hence

$$B_K(u,u) \ge (\pi^2 - K^2)\|u\|^2$$

and

$$B_K(u,u) \ge \frac{\pi^2 - K^2}{\pi^2} \|u'\|^2.$$

From the first inequality we easily conclude eq (2.4):

$$\|f\| = \sup_{v \in L^2(\Omega)} \frac{(f,v)}{\|v\|} \ge \frac{(f,u)}{\|u\|} \ge (\pi^2 - \alpha^2)\|u\|.$$

Equation (2.6) follows similarly.

Eq (2.5) follows from (2.4) by

$$\|u'\|^2 = (f,u) + K^2\|u\|^2 \le \|f\|\|u\| + \alpha^2\|u\|^2.$$

The proof is completed. ◁

Remark 2: The statement of the lemma holds also if u and v are choosen from a Hilbert subspace $V_h \subset V$. Indeed, since obviously $\min_{u\in V}(|u|_1/\|u\|) \leq \min_{u\in V_h}(|u|_1/\|u\|)$, the form B_K is strongly elliptic on the subspace and the same arguments apply.

2.2 Stability estimates for higher p

In part I we proved stability estimates for L^2- and H^{-1}-data. We now generalize these properties in two directions. Namely, we first consider data of higher (than L^2) regularity and prove that for $l \geq 2$ the solution norm $|u|_{l+1}$ is bounded by $k^{l-1}\|f\|_{l-1}$.

We then show a dual result, i.e. we bound $|u|_1$ by $\|f^{-m}\|$. In this case we consider data of a specific type - 'bubble' data - for which the integrals vanish at the boundaries of Ω. The sense of this assumption will become clear in the error analysis.

Theorem 2.1 (Continuous stability): *Let f be the data and u the solution of the BVP (1.1-1.3). Assume, for $l > 1$, $f(x) \in H^{l-1}(0,1)$. Then $u \in H^{l+1}(0,1)$ and the estimate*

$$|u|_{l+1} \leq C_s(l)k^{l-1}\|f\|_{l-1} \tag{2.7}$$

holds for a positive constant $C_s(l) \leq Dl$, where D does not depend on k and l.

Remark 3: Except for the dependence on k (and l), the statement is similar to the well known regularity result for the Laplace equation (cf. [Sch, p.52/53]).

Proof : Let us first consider the case $l = 2$. We have to prove $|u|_3 \leq Ck\|f\|_1$. We start from the Green's function representation of u (see part I for details):

$$u(x) = \int_0^1 G(x,s)f(s)ds \tag{2.8}$$

where

$$G(x,s) = \frac{1}{k}\begin{cases} \sin kxe^{iks}; & 0 \leq x \leq s \\ \sin kse^{ikx}; & s \leq x \leq 1 \end{cases}. \tag{2.9}$$

By partial integration,

$$u(x) = [H(x,s)f(s)]_{s=0}^{s=1} - \int_0^1 H(x,s)f'(s)ds \tag{2.10}$$

with

$$H(x,s) := \int G(x,s)ds = \frac{1}{k^2}\begin{cases} i\sin kxe^{iks}+1; & 0 \le x \le s \\ \cos kse^{ikx}; & s \le x \le 1 \end{cases}. \tag{2.11}$$

For any fixed s (or x, resp.), $H(x,s)$ is a H^2-function of x (or s, resp.). In the boundary points, $H(x,0)$ and $H(x,1)$ are smooth (C^∞) functions. We now estimate

$$|u(x)| \le |H(x,0)||f(0)| + |H(x,1)||f(1)| + \sup_{x,s}|H(x,s)|\|f'\|.$$

From eq (2.11) we have directly

$$\forall x,s: \quad |H(x,s)| \le \frac{2}{k^2}$$

and with

$$\forall s: \quad |f(s)| \le \sqrt{2}\|f\|_1$$

(for the proof see, e.g., [BKS]) we get

$$\|u\| \le \frac{2}{k^2}\left(1+2\sqrt{2}\right)\|f\|_1. \tag{2.12}$$

For an estimate of the derivatives of u, differentiate in eq (2.10) to obtain

$$u'(x) = [H_x(x,s)f(s)]_{s=0}^{s=1} - \int_0^1 H_x(x,s)f'(s)ds \tag{2.13}$$

and from this and differentiation w.r. to x in (2.11),

$$|u|_1 \le \frac{1}{k}\left(1+2\sqrt{2}\right)\|f\|_1. \tag{2.14}$$

Similarily, since $H \in H^2(\Omega)$, we obtain differentiating (2.13)

$$|u|_2 \le \left(1+2\sqrt{2}\right)\|f\|_1. \tag{2.15}$$

Finally, since $u \in H^3(\Omega)$, the differential equation $u''' + k^2u' = f'$ holds (at least in the weak sense). Hence

$$|u|_3^2 \le k^4|u|_1^2 + 2k^2|u|_1|f|_1 + |f|_1^2.$$

and with (2.14) we obtain

$$|u|_3^2 \le c^2k^2\|f\|_1^2 + 2ck\|f\|_1^2 + \|f\|_1^2$$

or, equivalently,

$$|u|_3 \leq C\,k\,\|f\|_1 \tag{2.16}$$

which proves the statement for $l = 2$.

For higher l we proceed analogously. First we integrate in eq (2.8) by parts $(l-1)$ times:

$$\begin{aligned} u(x) &= \left[G^{(-1)}(x,s)f(s)\right]_{s=0}^{s=1} - \left[G^{(-2)}(x,s)f'(s)\right]_{s=0}^{s=1} \pm \cdots \\ &\quad +(-1)^{l-2}\left[G^{(-l+1)}(x,s)f^{(l-2)}(s)\right]_{s=0}^{s=1} \\ &\quad +(-1)^{l-1}\int_0^1 G^{(-l+1)}(x,s)f^{(l-1)}(s)ds \end{aligned} \tag{2.17}$$

where $G^{(-j)}(x,s) = \int G^{-(j-1)}(x,t)dt$ with $G^{(0)}(x,s) := G(x,s)$ and appropriate integration constants for continuity at $x = s$.

For fixed x resp. s, the regularity is now $G^{(-j)}(x,s) \in H^{j+1}(0,1)$. At the boundaries we have again $G^{(-j)}(x,0), G^{(-j)}(x,1) \in C^\infty(0,1)$.

Therefore differentiation of $G^{(-j)}(x,s)$ w.r. to x is is well defined at most l times. Hence for $j = 1,\ldots,l$,

$$\begin{aligned} \left|u^{(j)}(x)\right| &\leq \left|G^{(j-1)}(x,0)\right||f(0)| + \left|G^{(j-1)}(x,1)\right||f(1)| + \ldots + \\ &\quad +\left|G^{(j-l+1)}(x,0)\right|\left|f^{(l-2)}(0)\right| + \left|G^{(j-l+1)}(x,1)\right|\left|f^{(l-2)}(1)\right| + \\ &\quad +\left|\int_0^1 G^{(j-l+1)}(x,s)f^{(l-1)}(s)ds\right|. \end{aligned}$$

The data is bounded in any point by

$$\forall x \in [0,1],\ \forall j: \quad \left|f^{(j)}(x)\right| \leq \sqrt{2}\|f^{(j)}\|_1 \leq \sqrt{2}\|f\|_{j+1}.$$

Generalizing eq (2.11), the integrals of the Green's function can be written in the form

$$G^{(-m)}(x,s) = k^{-(1+m)}\varphi(x,s) + k^{-2}P_{m-1}(x,s)$$

where $\varphi(x,s)$ is an oscillating part (with $\sup|\varphi| \leq 1$) and P_{m-1} is a sum of polynomials of degree $m-1$ in s and x, resp. Consequently, P_{m-1} is bounded on Ω and

$$\left|G^{(j-m)}(x,s)\right| \leq \begin{cases} C_1(j,m)k^{-2} + C_2(j,m)k^{j-m-1} & \text{if } j < m \\ k^{j-m-1} & \text{if } j \geq m \end{cases}.$$

In particular, for $j = l-1$ there exists a constant $C_3(l) = \max_{j,m}(C_1(j,m), C_2(j,m))$ s.t.

$$\left|u^{(l-1)}(x)\right| \leq C_3 \left(2\left(k^{l-3} + k^{l-4} + \ldots + k^{-1}\right) + k^{-1}\right) \sqrt{2}\, \|f\|_{l-1}.$$

Similarly, for $j = l$ there exists $C_4(l)$ s.t.

$$\left|u^{(p)}(x)\right| \leq C_4 \left(2\left(k^{p-2} + k^{p-3} + \ldots + 1\right) + 1\right) \sqrt{2}\, \|f\|_{p-1}.$$

Hence for $k > 1$ and $l > 2$ there are constants $C_5(l)$ and $C_6(l)$ not depending on k s.t.

$$|u|_{p-1} \leq C_5 k^{p-3} \|f\|_{p-1} \tag{2.18}$$

$$|u|_p \leq C_6 k^{p-2} \|f\|_{p-1} \tag{2.19}$$

hold. By their definition, C_5 and C_6 are of order l.

Then, finally,

$$\begin{aligned} |u|_{l+1}^2 &= \int_0^1 \left(f^{(l-1)} - k^2 u^{(l-1)}\right)^2 dx \\ &\leq k^4 |u|_{l-1}^2 + 2k^2 |u|_{l-1} |f|_{l-1} + |f|_{l-1}^2 \\ &\leq \left(k^4 C_5^2 k^{2l-6} + 2\,k^2 C_5 k^{l-3} + 1\right) \|f\|_{l-1}^2 \end{aligned}$$

and the statement readily follows. The proof is completed. $\lhd$

We now proceed to a second stability result that is dual to the first one, i.e. we will bound lower solution norms by negative norms of the data. We employ two auxiliary problems:

1. Consider the 2nd-order Dirichlet BVP on $\Omega = (0,1)$.

$$w'' - k^2 w = -g \tag{2.20}$$

$$w(0) = w(1) = 0 \tag{2.21}$$

and the associated variational problem: for $g \in H^1(0,1)'$ find $u \in H_o^1(0,1)$ s.t.

$$\forall v \in H_o^1(0,1): \quad B_1(w,v) = \int_0^1 w'v' + k^2 \int_0^1 wv = \int_0^1 gv = (g,v) \tag{2.22}$$

2. Consider the 4th-order Dirichlet BVP

$$w'''' + k^4 w = g \tag{2.23}$$
$$w(0) = w(1) = w'(0) = w'(1) = 0 \tag{2.24}$$

and the associated variational problem: for $g \in H^2(0,1)'$ find $u \in H_o^2(0,1)$ s.t.

$$\forall v \in H_o^2(0,1): \quad B_2(w,v) = \int_0^1 w''v'' + k^4 \int_0^1 wv = \int_0^1 gv = (g,v) \tag{2.25}$$

The forms B_1 and B_2 are coercive, hence eqs (2.22) and (2.25) have unique solutions.

Lemma 2.3 *Let u_1 and u_2 be the solutions of $B_1(w,v) = (g,v)$ and $B_2(w,v) = (g,v)$, respectively.*
Then for u_1

$$\|u_1'\| \leq \|g^{(-1)}\| \tag{2.26}$$

and for u_2

$$\|u_2''\| \leq \|g^{(-2)}\| \tag{2.27}$$
$$k\|u_2'\| \leq \frac{1}{\sqrt{2}}\|g^{(-2)}\| \tag{2.28}$$
$$k^2\|u_2'\| \leq \frac{1}{2}\|g^{(-1)}\| \tag{2.29}$$

hold.

Proof : Let $|||\cdot|||_1$ and $|||\cdot|||_2$ be the energy norms induced by the forms B_1 and B_2, resp. In these norms, the B-B-constant is trivially $\gamma = 1$ and hence

$$|||u_1|||_1 \leq |||g|||_1'$$

$$|||u_2|||_2 \leq |||g|||_2'.$$

It is then easy to see from the definitions of the various norms that

$$\|u_1'\| \leq |||u_1|||_1 \leq |||g|||_1' \leq \|g\|_1' = \|g^{(-1)}\|$$

$$\|u_2''\| \leq |||u_2|||_2 \leq |||g|||_2' \leq \|g\|_2' = \|g^{(-2)}\|$$

hold. This proves (2.26) and (2.27).

To show eq (2.29), let $u \in H_o^2(\Omega)$. From

$$0 \leq (\|u''\| - k^2\|u\|)^2$$

we conclude

$$2k^2\|u''\|\|u\| \leq B_2(u,u)$$

and, by Schwarz inequality and partial integration,

$$B_2(u,u) \geq 2k^2|(u,u'')| = 2k^2\|u'\|^2. \tag{2.30}$$

On the other hand, also by partial integration and Schwarz inequality,

$$B_2(u,u) \leq |(g^{(-1)},u')| \leq \|g^{(-1)}\|\|u'\|.$$

Hence

$$2k^2\|u'\|^2 \leq \|g^{(-1)}\|\|u'\|.$$

Cancelling $\|u'\|$ we obtain (2.29).

Finally, multiplying (2.30) by the trivial relation $B_2(u,u) \geq \|u''\|^2$ we have

$$2k^2\|u'\|^2\|u''\|^2 \leq B_2(u,u)^2. \tag{2.31}$$

But $B_2(u,u)$ is majorized by $\|g^{(-2)}\|\|u''\|$, and (2.28) readily follows cancelling $\|u''\|^2$ and taking roots. This completes the proof. $\lhd$

Theorem 2.2 (A dual stability result): *Assume that, for integer $m \geq 1$, there is given a function $f \in L^2(\Omega)$ such that $f^{(-i)}(0) = f^{(-i)}(1) = 0$ for $i = 1,\dots,m$. Let $u \in H^1_{(o}(\Omega)$ be the solution to the VP (1.4) with data f.*

Then

$$|u|_1 \leq C_1 k^m \|f^{(-m)}\| + C_2\|f^{(-1)}\| \tag{2.32}$$

holds with C_1, C_2 independent of k.

Remark 4: The assumption on the data means that m integrals of f vanish in $x = 0$ (note that all integrals vanish at the endpoint by definition). Without this assumption we can just prove that

$$|u|_1 \leq C\, k\, \|f^{(-1)}\|$$

(see part I). In general, $|u|_1$ cannot be bounded by a term $C\|f^{(-1)}\|$ indepenently of k.

Proof : (1) For $m = 1$ the statement simplifies to

$$|u|_1 \leq Ck\|f^{(-1)}\|$$

and is directly obtained from the B-B-condition (Introduction, A4.) and general theory ([BA]).

(2) Let $m = 2$. The basic ingredience of the argument is the introduction of a smoother kernel to the Green's function representation $u(x) = (G(x,s), f(s))$ of the solution. We define

$$K(x,s) := G(x,s) - H(x,s)$$

where $H(x,s)$ is the Green's function to the first auxiliary problem (2.22). Per definition of Green's functions, the equations

$$H_{xx}(x,s) - k^2 H(x,s) = -\delta_s(x) \tag{2.33}$$

$$G_{xx}(x,s) + k^2 G(x,s) = -\delta_s(x) \tag{2.34}$$

hold (in the sense of distributions). Hence, after substraction,

$$K_{xx} = -k^2(G+H). \tag{2.35}$$

Since $G, H \in H^1(\Omega)$ (as functions of x for any fixed s) it follows that $K \in H^3(\Omega)$. We have constructed an equivalent integral representation of u as

$$u(x) = \int_0^1 K(x,s)f(s)ds + \int_0^1 H(x,s)f(s) := u_1(x) + u_2(x). \tag{2.36}$$

For estimation of $\|u_1'\|$ we integrate by parts

$$u_1'(x) = \int_0^1 K_x(x,s)f(s)ds = \int_0^1 K_{xss}(x,s)f^{(-2)}(s)ds,$$

(note that the boundary terms vanish due to the specific assumption on f) hence by Cauchy-Schwarz inequality

$$|u_1'(x)| \leq \|K_{xss}\|\|f^{(-2)}\|.$$

From eq (2.35) and the symmetry property of Green's functions, $K_{xss} = (G+H)_x$ and

$$\|K_{xss}\| \leq k^2 (\|G_x\| + \|H_x\|)$$

follows. It is straightforward to show that $\|G_x\|$ and $\|H_x\|$ are bounded independently of k and we thus have

$$|u_1'(x)| \le C\, k^2 \|f^{(-2)}\|. \tag{2.37}$$

To estimate u_2 we apply eq (2.26), Lemma 2.2, for $g = f$:

$$\|u_2'\| \le \|f^{(-1)}\|$$

which, together with eq (2.37) yields the statement, case 2.

(3) Consider the case $3 \le m \le 6$. In analogy to the previous step, we introduce a still smoother kernel to the integral representation of u. Let us prove the extreme case $m = 6$. We start from eq (2.36) and define

$$L(x,s) := K(x,s) - J(x,s)$$

where $J(x,s)$ is a Green's function with the same singularity as K. Then

$$\begin{aligned} u(x) &= \int_0^1 L(x,s)f(s)ds + \int_0^1 J(x,s)f(s) + \int_0^1 H(x,s)f(s) \\ &:= u_o(x) + u_1(x) + u_2(x). \end{aligned} \tag{2.38}$$

To find the appropriate J, add eqs (2.33) and (2.34):

$$(G+H)_{xx}(x,s) + k^2(G-H)(x,s) = -2\delta(x,s).$$

Multiplying this eq by $-k^2$ and substituting $K = G - H$, eq (2.35) leads to

$$K_{xxxx}(x,s) - k^4 K(x,s) = 2\,k^2\,\delta_s(x,s).$$

Hence if $J(x,s)$ is the Green's function to the second auxiliary problem with data $g(x) = 2\,k^2 f(x)$ then

$$L_{xxxx} = (K-J)_{xxxx} = k^4\,(K+J).$$

Since $K, J \in H^3(\Omega)$ it follows that $L \in H^7(\Omega)$. Thus, integrating by parts,

$$\begin{aligned} |u_o'(x)| &= \left| \int L_{xssssss}(x,s) f^{(-6)}(s)ds \right| \\ &= k^4 \left| \int (K+J)_{ssx}(x,s) f^{(-6)}(s)ds \right| \\ &= k^4 \left| \left(k^2 \int (G+H)_x(x,s) f^{(-6)}(s)ds + \int J_x f^{(-4)}(s)ds \right) \right| \end{aligned}$$

and therefore

$$\|u_o'\| \le Ck^6\|f^{(-6)}\| + k^4\|u_3'\|$$

where u_3 is a solution to the second auxiliary problem (2.25) with data $g = 2\,k^2 f^{(-4)}$. From Lemma 2.2 and Poincaré's inequality,

$$\|u_3'\| \le |u_3|_2 \le 4k^2\|f^{(-6)}\|,$$

hence

$$\|u_o'\| \le C_1\, k^6\|f^{(-6)}\|.$$

By definition (2.38), u_1 is the solution to the VP (2.25) with data $g = 2\,k^2 f$ and u_2 solves the VP (2.22) with data f. Then it follows from Lemma 2.2. that

$$\|u_1'\| \le \frac{c}{k^2}\,\|g^{(-1)}\| = C\,\|f^{(-1)}\|$$

and

$$\|u_2'\| \le \|f^{(-1)}\|$$

so that finally u can be estimated as

$$|u|_1 \le C_1\, k^6\,\|f^{(-6)}\| + C_2\|f^{(-1)}\|$$

which proves the statement for $m = 7$.

For $m = 4 \ldots m = 6$ the argument is completely analogous, the only difference being the number of partial integrations in the representation of u_o. For $m \ge 7$, we introduce further smootheners and proceed in the same manner. The proof is completed. $\lhd$

3 Finite element solution

In this section we analyze the h-version of the Galerkin finite element method with approximation order p. We first identify the finite dimensional approximation space $V_h \subset H^1_{(o}(\Omega)$ and prove two properties of the best approximation in H^1-seminorm. After an outline of the finite element solution procedure (subsection 3.2) we turn to the analysis of a h-p Galerkin finite element method for the numerical solution of the model problem. In subsection 3.3. we prove a result on the phase difference between the exact and the numerical solutions. In subsection 3.4. we give several stability estimates for the numerical solution u_{fe}. We prove a discrete inf-sup-condition for general p and conclude stability of u_{fe} w.r.t. to L^2-data, measured in L^2- or H^{-1}-norm,

respectively. We then proceed to the proof of a dual estimate in still lower norms, given for specific 'bubble' data as it is encountered in the error estimation for $p > 1$. All stability constants are explicitely computed w.r. to the wavenumber k. The analysis of the finite element method is concluded with error estimates (subsection 3.5.).

3.1 The approximation space V_h

Assume that the solution domain Ω has been uniformly divided into n disjunct intervals $\Delta_i = (x_{i-1}, x_i)$ called finite elements. Let $h = x_i - x_{i-1}$ (stepwidth). The set of nodal points

$$X_h = \{0 = x_o, x_1, \ldots, x_n = 1\} \tag{3.1}$$

will be called finite element mesh. We will consider mesh functions defined on X_h and refer to them by subscript h; for the nodal value of a mesh function u_h in a node $x_i \in X_h$ we will write shortly $u_i := u_h(x_i)$.

Let $p > 0$ be integer and Δ a finite element. We denote by $S^p(\Delta)$ the linear space of all polynomials with domain of definition $\bar{\Delta}$ and degree $\leq p$. For given mesh, we define the space of piecewise polynomial, continuous functions

$$V_h := S_h^p(0,1) := \left\{ s \in H^1_{(o}(\Omega),\;\; s(x)\,|_{\Delta_i} \in S^p(\Delta_i),\;\; i = 1 \ldots n \right\}.$$

Thus by definition $V_h \subset H^1_{(o}(0,1)$. If not stated otherwise, we will assume that V_h is equipped with the H^1-seminorm.

Within the elements we introduce the local coordinate ξ by linear mapping $\Delta_i \to I = (-1,1)$. The polynomials in $S^p(I)$ are then written as linear combinations of the *nodal* shape functions

$$N_1(\xi) = \frac{1-\xi}{2}\ ,\ N_2(\xi) = \frac{1+\xi}{2}\ ;\ -1 \leq \xi \leq 1$$

and (if $p > 1$) the *internal* shape functions

$$N_l(\xi) = \phi_{l-1}(\xi)\ ;\ l = 3, 4, \ldots, p+1$$

where ϕ_l is written in terms of the Legendre polynomials P_j

$$\phi_l(\xi) = \sqrt{\frac{2l-1}{2}} \int_{-1}^{\xi} P_{l-1}(t)dt$$

(see [SB, pp. 38/39]). The internal shape functions vanish at the element boudaries, forming the subspace $S_o^p(I) = \text{span}\,\{N_3, N_4, \ldots, N_{p+1}\} \subset S^p(I)$.

Let $\Delta_i = (x_{i-1}, x_i)$ be a finite element of length h. Denote by $(\cdot,\cdot)_\Delta$ the L^2 inner product on Δ and by $\|\cdot\|_\Delta$ the induced local L^2-norm. Similarly we use the notation $\|\cdot\|_I$ on $I = (-1,1)$. On $S^p(I)$, an inverse inequality (in p) is given by the well known Markov Theorem[2] stating for $y \in S^p(I)$ the inequality

$$\|y'\|_I \le C_{inv}(p)\, p^2 \|y\|_I \tag{3.2}$$

with $C_{inv}(p) = (p+1)^2/(p^2\sqrt{2})$ [Be]. Hence $C_{inv}(p) \le 4\sqrt{2}$ and $C_{inv}(p) \to 1/\sqrt{2}$ for $p \to \infty$.

We now prove that for any $u \in H^{l+1}(\Omega)$ one can find a piecewise polynomial, nodally exact (on X_h) function $s \in V_h$ s.t.

- s is an optimal (in h and p) approximation of u in the H^1-seminorm and
- the integrals of s are nodally exact, quasioptimal approximations of the integrals of u.

Theorem 3.1 (Approximation in V_h): *Let l, p be integers with $1 \le l \le p$ and let $u \in H^{l+1}(0,1)$. There exists an $s \in V_h = S_h^p(0,1)$ s.t.*

1. *(nodally exact approximation)*

$$\forall x_i \in X_h: \quad s^{(m)}(x_i) = u^{(m)}(x_i), \quad m = -p+1, \dots, 0 \tag{3.3}$$

2. *(order of approximation)*

$$\|(u-s)^{(m)}\| \le C_a(l) C_a(-m) \left(\frac{h}{2p}\right)^{l-m+1} |u|_{l+1}, \quad m = -p+1, \dots, 1 \tag{3.4}$$

where C_a satisfies:

1. *$C_a(-1) = 1$ (formal definition)*
2. *$C_a(0) = 1$,*
3. *C_a decreases for $0 \le l \le \sqrt{p}$,*
4. *C_a increases for $l > \sqrt{p}$ and*

[2] This inequality is usually given in the L^∞-norm (cf., egs., [N, p. 124]). We use an L^2-variant [Be].

5.

$$C_a(p) = \left(\frac{e}{2}\right)^p (\pi p)^{-1/4} \tag{3.5}$$

is the maximum of $C_a(l)$ *over* $l \in \{0, 1, \dots, p\}$.

Remark 5: With respect to the stepwidth h, the estimate (3.4) is the standard approximation result of finite element theory (see, e.g., [Sch, pp. 46–49]).

Remark 6: For $l = 0, 1$ the statements are proven in [BKS]. The following argument is a generalization of this proof.

Proof : We start on the local level. Let Δ_i be a finite element and let $I = (-1, 1)$. We write $u'(\xi) \in H^l(I) \subset L^2(I)$ as

$$u'(\xi) = \sum_{i=0}^{\infty} a_i P_i(\xi)$$

where $P_i(\xi)$ are the Legendre polynomials of order i and equality is understood in the L^2-sense. Set

$$s'(\xi) := \sum_{i=0}^{p-1} a_i P_i(\xi)$$

and define the integrals ($i = 0, 1, 2, \dots$):

$$u^{(-i)}(\xi) = u^{(-i)}(1) - \int_\xi^1 u^{(-i+1)}(\tau) d\tau \tag{3.6}$$

$$s^{(-i)}(\xi) = u^{(-i)}(1) - \int_\xi^1 s^{(-i+1)}(\tau) d\tau. \tag{3.7}$$

We will now prove that (3.3) holds. Let first $i = 0$, then from eqs (3.6, 3.7) we have trivially $u(1) = s(1)$. Further, by definition,

$$u(\xi) = u(1) - \int_\xi^1 u'(\tau)d\tau = u(1) - \sum_{j=0}^{\infty} a_j \int_\xi^1 P_j(\tau)d\tau = u(1) - 2a_o$$

$$= u(1) - \int_{-1}^1 s'(t)dt = u(1) - \sum_{j=0}^{p-1} a_j \int_\xi^1 P_j(\tau)d\tau = s(-1).$$

Now we integrate $u'(\xi)$, using the well known relation

$$P_i(\tau) = (P'_{i-1}(\tau) - P'_{i+1}(\tau))/(2i+1)$$

to obtain

$$u(\xi) = u(1) + a_o(P_1(\xi) - P_o(\xi)) + \sum_{i=1}^{\infty} a_i \frac{P_{i+1}(\xi) - P_{i-1}(\xi)}{2i+1}.$$

Integrating once more (we write $U := u^{(-1)}$),

$$U(-1) = U(1) - \int_{-1}^{1} u(\tau)d\tau = U(1) - 2u(1) - 2a_o - \frac{2a_1}{3}.$$

Obviously, the same result is obtained from the integration of the polynomial $s(\xi)$ since only the coeffient of P_o influences the result of integration over the whole interval I. By similar argument we conclude that integration of the polynomial s on the one and the function u on the other hand leads to the same result exactly $p-1$ times. Indeed, by replacing repeatedly $P_i(\tau) = (P'_{i-1}(\tau) - P'_{i+1}(\tau))/(2i+1)$, we see that with the ith succesive integration of $u(\xi)$ or $s(\xi)$ the coefficient a_i enters the set of coefficients multiplying P_o. Since the norms of $u^{(i)}$ and $s^{(i)}$ depend only on the coefficient of P_o, both norms are equal until P_o is multiplied by a_{p-1}, i.e. in general $u^{(-p+1)}(\xi) = s^{(-p+1)}(\xi)$ and $u^{(-p)}(\xi) \neq s^{(-p)}(\xi)$. Thus nodal exactness, eq (3.3), is proved on an arbitrary element and hence it holds globally.

Let us now prepare the proof of estimate (3.4). With above definitions, the error of approximation is

$$e'(\xi) := u'(\xi) - s'(\xi) = \sum_{i=p}^{\infty} a_i P_i(\xi)$$

and from the orthogonality property of the Legendre polynomials we have

$$\|e'\|^2 = \sum_{i=p}^{\infty} \frac{2}{2i+1} a_i^2. \tag{3.8}$$

It can be proven (see [BKS], chapter 3) that s' is the best L^2-approximation to u' on I and the estimate

$$\|u' - s'\| \leq \frac{C_a(l)}{p^l} |u|_{l+1} \tag{3.9}$$

holds for $0 \leq l \leq p$, where the constant C_a has the properties 2. - 5. given in the theorem.

Integrating the error e', we get

$$e(\xi) = \int_\xi^1 (u'(t) - s'(t))\, dt = \sum_{i=p}^{\infty} a_i \int_\xi^1 P_i(t)dt = -\sum_{i=p}^{\infty} \frac{a_i}{2i+1} (P_{i+1}(\xi) - P_{i-1}(\xi))$$

After reordering,

$$e(\xi) = \sum_{i=p+1}^{\infty} b_i P_i(\xi) + \frac{a_p}{2p+1} P_{p-1}(\xi) + \frac{a_{p+1}}{2p+3} P_p(\xi)$$

with

$$b_i = \frac{a_{i+1}}{2i+3} - \frac{a_{i-1}}{2i-1}$$

and the norm is

$$\|e\|^2 = \sum_{i=p+1}^{\infty} \frac{2}{2i+1} b_i^2 + \frac{a_p^2}{(2p+1)^2} \frac{2}{2p-1} + \frac{a_{p+1}^2}{(2p+3)^2} \frac{2}{2p+1} \tag{3.10}$$

We apply the relation $(a-b)^2 \leq 2a^2 + 2b^2$ to obtain (for $i \geq p+1$)

$$b_i^2 \leq \frac{2a_{i-1}^2}{(2i-1)^2} + \frac{2a_{i+1}^2}{(2i+3)^2}$$

and thus

$$\sum_{i=p+1}^{\infty} b_i^2 \frac{2}{2i+1} \leq \frac{1}{2p^2} \sum_{i=p}^{\infty} a_i^2 \frac{2}{2i+1} + \frac{1}{2p^2} \sum_{i=p+2}^{\infty} a_i^2 \frac{2}{2i+1}$$

holds. Taking now into account the second and third member in the r.h.s. of eq (3.10) we get

$$\|e\|^2 \leq \frac{1}{p^2} \sum_{i=p}^{\infty} a_i^2 \frac{2}{2i+1}$$

and hence

$$\|e\| \leq \frac{1}{p} \|e'\|. \tag{3.11}$$

From eq (3.9) it then follows that

$$\|e\| \leq \frac{C_a(l)}{p^{l+1}} |u|_{l+1} \tag{3.12}$$

holds for $1 < l \leq p$.

Let us conclude the local analysis showing an orthogonality property for e. Since s' is the L^2-projection of u' on $S^{p-1}(I)$,

$$\int_{-1}^{1}(u'(\xi)-s'(\xi))\xi^m d\xi = 0 \tag{3.13}$$

holds for $m=0,1,\ldots,p-1$. We claim that $e(\xi)=\int_\xi^1(u'(t)-s'(t))dt$ is orthogonal to $S^{p-2}(I)$.

Indeed, for $l \geq 0$ we compute

$$\begin{aligned}\int_{-1}^{1} e(\xi)\xi^m d\xi &= \int_{-1}^{1}(\int_{\xi}^{1}(u'(t)-s'(t))dt)\xi^m d\xi \\ &= \int_{-1}^{1}((u'(t)-s'(t))\int_{-1}^{t}\xi^m d\xi)dt \\ &= \frac{1}{m+1}\int_{-1}^{1}(u'(t)-s'(t))(t^{m+1}+1)dt.\end{aligned}$$

which, together with eq (3.13), proves that $e \perp S^{p-2}(I)$.

This completes the local analysis. By back-transform $I \to \Delta$ and summation over the elements we conclude eq (3.4) for H^1- and L^2-norm , i.e. cases $m=1,0$ in eq (3.4).

It remains to prove eq (3.4) for dual norms. We apply a standard argument [Sch]. By definition, for $m \geq 1$,

$$\|e\|_{-m} = \sup_{v \in H^m_{(o}} \frac{(e,v)}{|v|_m}.$$

Let $P^m v \in S_h^m(\Omega)$ be the L^2-projection of $v \in H^m_{(o}$ on $S_h^m(\Omega)$. Then by orthogonality, as proven in step 1,

$$\|e\|_{-m} = \sup_{v \in H^m_{(o}} \frac{(e, v-P^{m-1}v)}{|v|_m}$$

holds for $1 \leq m \leq p-1$. Applying Schwartz inequality and eq (3.12) we conclude for $1 \leq l \leq p$ and $1 \leq m \leq p-1$, the estimate:

$$\|e\|_{-m} \leq C_a(l)\left(\frac{h}{2p}\right)^{l+1}|u|_{l+1}\,C_a(m)\left(\frac{h}{2p}\right)^{m}\frac{|v|_m}{|v|_m} \leq C(l,m)\left(\frac{h}{2p}\right)^{l+m+1}|u|_{l+1}$$

where

$$C(l,m) = C_a(l)C_a(m) \leq \left(\frac{e}{2}\right)^{2p}(\pi p)^{-1/2}.$$

This completes the proof of Theorem 3.1. $\triangleleft$

3.2 The finite element method

Let $V_h \subset H^1_{(o}(\Omega) =: V$ be the approximation space introduced in the previous subsection. As usual, function $u \in V_h$ is called the finite element solution of the VP (1.4) if

$$\forall v \in V_h : \quad B(u_{fe}, v) = (f, v). \tag{3.14}$$

The approximation space $V_h = S^p_h(\Omega$ can be written as a direct sum of two subspaces, namely,

$$S^p_h(\Omega) = S^1_h(\Omega) \oplus S^p_o(\Omega),$$

where S^1_h is the space of continuous, piecewise linear functions and

$$S^p_o(\Omega) = \bigoplus_{j=1}^{n} S^p_o(\Delta_j)$$

with

$$S^p_o(\Delta_j) = \mathrm{span}\{N^j_3, \ldots, N^j_{p+1}\}.$$

Here, Δ_j are the finite elements, hence $S^p_o(\Delta_j)$ are local spaces of "bubble" polynomials.

Writing now $u_{fe} = u_h + u_p, v = v_h + v_p$, where $u_h, v_h \in S^1_h(\Omega)$ and $u_p, v_p \in S^p_o(\Omega)$, we have from eq (3.14)

$$\forall v_h \in S^1_h(\Omega) : \quad B(u_{fe}, v_h) = (f, v_h). \tag{3.15}$$

and

$$\forall v_p \in S^p_o(\Omega) : \quad B(u_{fe}, v_p) = (f, v_p). \tag{3.16}$$

From eq (3.15) we conclude

$$\forall v_h \in S^1_h(\Omega) : \quad B(u_h, v_h) = (f + k^2 u_p, v_h). \tag{3.17}$$

On the other hand, eq (3.16), inforce of $S^p_o(\Omega) = \oplus^n_1 S^p_o(\Delta)$, decouples into n independent local systems

$$\forall w \in S^p_o(\Delta) : \quad B_\Delta(u_p, w) = (f + k^2 u_h, w)_\Delta, \tag{3.18}$$

where B_Δ is the restriction of the form B to the finite element Δ. By formally solving these equations, we express u_p in terms of f and u_h. The result can be inserted into (3.17), giving rise to

$$\tilde{B}(u_h, v_h) = (\tilde{f}, v_h). \tag{3.19}$$

Discretising this equation by standard approach, we obtain the linear system

$$[L_h]\{u_h\} = \{r_h\}, \tag{3.20}$$

where $[L_h]$ is a $(n \times n)$ matrix, usually called the condensed stiffness matrix, and $\{u_h\} = u_{fe}|_{X_h}$ is the vector of nodal values of the finite element solution on the mesh X_h. The piecewise linear part of u_{fe} is determined by eq (3.20), provided $[L_h]$ is nonsingular. The internal part u_p can then be found locally by eqs. (3.18).

For the sake of further analysis, we outline below the details of the solution procedure:

Step 1 (Local approximation and static condensation): On any element Δ_j, the trial function u and the test function v are written as scalar products of shape functions $\left\{N_1^j, N_2^j, \ldots, N_{p+1}^j\right\}$ and the vectors of unknown coefficients $\{a^j\} = \left\{a_1^j, a_2^j, \ldots, a_{p+1}^j\right\}^T$ and $\{b^j\} = \left\{b_1^j, b_2^j, \ldots, b_{p+1}^j\right\}^T$, respectively. Identifying $a_1^j = u(x_{j-1}), a_2^j = u(x_j)$ and $b_1^j = v_{j-1}, b_2^j = v_j$, we have $u_h|_{\Delta_j} = a_1^j N_1^j + a_2^j N_2^j$. The condition that u_{fe} be the solution of the VP (1.4) for all $v \in V_h$ leads locally (i.e. on Δ_j) to

$$\{\bar{b}^j\}^T [B^j]\{a^j\} = \{\bar{b}^j\}^T\{r^j\} \tag{3.21}$$

with the $(p+1) \times (p+1)$ square matrix $(l, m = 1, \ldots, p+1)$

$$[B^j] = \left[\{\int_{\Delta_j} N_l^j(x)' N_m^j(x)' dx - k^2 \int_{\Delta_j} N_l^j(x) N_m^j(x) dx - ikN_l^j(1)N_m^j(1)\}\right] \tag{3.22}$$

and the right hand side

$$\{r^j\} = \left\{(f(x), N_l^j(x))_{\Delta_j}, l = 1, \ldots, p+1\right\}^T.$$

Now, decomposing

$$[B^j] = \begin{bmatrix} \left[B_{11}^j\right] & \left[B_{12}^j\right] \\ \left[B_{21}^j\right] & \left[B_{22}^j\right] \end{bmatrix} \tag{3.23}$$

where $\left[B_{11}^j\right]$ is the left upper 2×2-submatrix of $[B^j]$, and assuming for the moment that $\left[B_{22}^j\right]$ is nonsingular, we define

$$[CB^j] = \left[B_{11}^j\right] - \left[B_{12}^j\right]\left[B_{22}^j\right]^{-1}\left[B_{21}^j\right]. \tag{3.24}$$

Then, by local variation of $\{b_3^j, \ldots b_{p+1}^j\}^T$, we find – cf. eq (3.18) –

$$\{v_{j-1}\ v_j\}\left[CB^j\right]\left\{\begin{array}{c} u_{j-1} \\ u_j \end{array}\right\} = \{v_{j-1}\ v_j\}\left\{\begin{array}{c} \tilde{r}_{j-1} \\ \tilde{r}_j \end{array}\right\}, \tag{3.25}$$

where

$$\left\{ \begin{array}{c} \tilde{r}_{j-1} \\ \tilde{r}_j \end{array} \right\} = \left\{ \begin{array}{c} r_1^j \\ r_2^j \end{array} \right\} - \left[B_{12}^j\right] \left[B_{22}^j\right]^{-1} \left\{ \begin{array}{c} r_3^j \\ \vdots \\ r_{p+1}^j \end{array} \right\}. \tag{3.26}$$

On uniform mesh, the local matrices $[CB^j]$ are identical on all elements and can be written in the form

$$[CB] = \begin{bmatrix} S_p(K) & T_p(K) \\ T_p(K) & S_p(K) \end{bmatrix} \tag{3.27}$$

where $S_p(K)$ and $T_p(K)$ are rational polynomial functions of $K = kh/2$.

Remark 7: The analogy to the h-version with piecewise linear approximation is given by the following consideration. On Δ_j, the homogeneous h-p-finite element solution is written as

$$u_{fe}(x) = u_{fe}(x_{j-1})N_1^p(x) + u_{fe}(x_j)N_2^p(x) \tag{3.28}$$

where the condensed shape functions $N_j^p \in S^p(\Delta_j)$ are local variational solutions to the homogeneous Helmholtz equation with appropriate boundary conditions.

The local stiffness matrix is written as[3]

$$[CB^j] = \begin{bmatrix} B(N_1^p, N_1^p) & B(N_1^p, N_2^p) \\ & \\ B(N_2^p, N_1^p) & B(N_2^p, N_2^p) \end{bmatrix}$$

which is again the formal analogon to the h-version (cf. part I).

Step 2 (Global assembling and solution for u_h): Enforcing continuity of the test functions in the nodal points of X_h we obtain the set of linear equations (3.20). The discrete operator L_h is an $n \times n$ tridiagonal matrix:

$$[L_h] = \begin{bmatrix} 2S_p(K) & T_p(K) & & \\ T_p(K) & 2S_p(K) & T_p(K) & \\ & & \ddots & \\ & T_p(K) & 2S_p(K) & T_p(K) \\ & & T_p(K) & S_p(K) - iK \end{bmatrix}, \tag{3.29}$$

[3]see next subsection for the proof

the global right hand side vector is

$$R_h = \left\{ \begin{array}{c} \tilde{r}_1 \\ \vdots \\ \tilde{r}_{j-1} + \tilde{r}_j \\ \vdots \\ \tilde{r}_n \end{array} \right\} \tag{3.30}$$

and $K = kh/2$ is a measure for the number of elements per wavelength (see part I, remark 8).

Step 3 (Local "decondensation"): The discrete equivalent of eqs (3.18)

$$\left[B_{22}^j\right] \left\{ \begin{array}{c} a_3^j \\ \vdots \\ a_{p+1}^j \end{array} \right\} = \left\{ \begin{array}{c} r_3^j \\ \vdots \\ r_{p+1}^j \end{array} \right\} - \left[B_{21}^j\right] \left\{ \begin{array}{c} u_{j-1} \\ u_j \end{array} \right\} \tag{3.31}$$

can be inverted - provided $\left[B_{22}^j\right]$ is regular - to determine locally the bubble modes of the finite element solution.

3.3 Discrete Green's function and discrete wavenumber

The nodal values of the finite element solution on X_h are found from the tridiagonal linear system (3.20). On uniform mesh, this system consists of formally identical difference stencils

$$T_p(K)u_h(x_{j-1}) + 2S_p(K)u_h(x_j) + T_p(K)u_h(x_{j+1}).$$

Equating these stencils to zero we find the homogeneous solutions

$$y_{h1} = \exp(ik'x_h), \quad y_{h2} = \exp(-ik'x_h) \tag{3.32}$$

where the parameter k' is determined as a function of k, h and p by

$$\cos(k'h) = -\frac{S_p(K)}{T_p(K)}. \tag{3.33}$$

The discrete wavenumber k' is, in general, different from k, causing a phase difference between the finite element and exact solutions.

The discrete finite element solution $\{u_h\}$ can be written as

$$u_h(x_i) = h \sum_{j=1}^{n} G_h(x_i, s_j) R_h(s_j) \tag{3.34}$$

with the discrete Green's function (cf. part I)

$$G_h(k', x_i, s_j) = \frac{1}{\sin k'h} \begin{cases} \sin k'x_i\,(A \sin k's_j + \cos k's_j) & x \leq s \\ \sin k's_j(A \sin k'x_i + \cos k'x_i) & s \leq x \leq 1 \end{cases} . \tag{3.35}$$

Note that the Green's function does not depend directly on p. The dependence occurs only implicitely through the parameter k'. This means that the estimates of G that are given in part I for piecewise linear approximation carry over to higher p without modification - except for the estimation of k' in terms of k, h and p.

Theorem 3.2 (Phase difference): *Let $p \geq 1$ and k' be the parameter in the fundamental system (3.32) of the set of linear equations (3.20).*

Then, if $hk < 1$,

$$|k' - k| \leq k\, C \left(\frac{C_a(p)}{2}\right)^2 \left(\frac{hk}{2p}\right)^{2p} \tag{3.36}$$

where k is the exact wavenumber, C_a is the approximation constant from Theorem 3.1. and C does not depend on k, h and p.

Remark 8: The phase difference between the exact and the finite element solution has been extensively studied in [HH] (for $p = 1$) and [TP1] (for $p = 1, 2, 3$). As a conlusion from numerical experiments, the statement of the theorem has been induced in w.r.t. to k and h in [TP1]. We now prove that with increasing p the phase difference is also going down with a factor $\left(\frac{e}{2}\right)^{2p} (\pi p)^{-1/2} (2p)^{-2p}$, i.e. the improvement with higher approximation is still more significant than it was assumed in above named references.

We will give the proof of this theorem after the following preliminary discussion: First we observe that any homogeneous solution to the Helmholtz equation can be written on each inner element $\Delta_j \subset \Omega$ as

$$u(x) = u^1 t_1(x) + u^2 t_2(x) \tag{3.37}$$

where $u^1 := u(x_{j-1}), u^2 := u(x_j)$ are the nodal values of u on X_h, whereas t_1, t_2 satisfy

$$t'' + k^2 t = 0 \quad \text{on} \quad \Delta_j \tag{3.38}$$

with inhomogeneous local Dirichlet data

$$t_1(x_{j-1}) = 1, \quad t_1(x_j) = 0 \tag{3.39}$$

or

$$t_2(x_{j-1}) = 0, \quad t_2(x_j) = 1, \tag{3.40}$$

resp.

By discrete evaluation of the VP (1.4) in the nodal points of X_h we find that, for $j = 1, \dots, n-1$,

$$T_o(K)u(x_{j+1}) + 2S_o(K)u(x_j) + T_o(K)u(x_{j-1}) = 0$$

holds with

$$\begin{aligned} T_o(K) &= B(t_1, t_2) = B(t_2, t_1) \\ 2S_o(K) &= B(t_1, t_1) + B(t_2, t_2). \end{aligned}$$

The fundamental solutions are

$$z_{h1} = \exp(ikx_h), \quad z_{h2} = \exp(-ikx_h)$$

and

$$\cos(kh) = -\frac{S_o(K)}{T_o(K)}$$

holds.

Second, writing the finite element solution on Δ_j as

$$u_{fe}(x) = u^1_{fe}N^p_1(x) + u^2_{fe}N^p_2(x)$$

where $N^p_1, N^p_2 \in S^p(\Delta_j)$ are approximate solutions to eq (3.38) with boundary conditions ((3.39) or (3.40), resp., we have

$$T_p = B(N^p_1, N^p_2) = B(N^p_2, N^p_1) \tag{3.41}$$

$$2S_p = B(N^p_1, N^p_1) + B(N^p_2, N^p_2). \tag{3.42}$$

To see this, let us analyze the BVP's (3.38, 3.39) and (3.38, 3.40) on $I = (-1, 1)$. After linear transformation $\Delta_j \to I$ we arrive at

$$t''(\xi) + K^2 t(\xi) = 0. \tag{3.43}$$

We assume that $K \leq \alpha < 2\pi$. The boundary conditions are

$$t(-1) = 1, \quad t(1) = 0 \tag{3.44}$$

or

$$t(-1) = 0, \quad t(1) = 1 \tag{3.45}$$

resp.

To formulate an equivalent variational problem, we write the admissible functions as

$$t = N_1 + \phi_1 \quad \text{or} \quad t = N_2 + \phi_2$$

resp., where $\phi_i \in H_o^1(I)$ and N_1, N_2 are the linear shape functions. The objective is then to find $\phi_i^o \in H_o^1(I)$ such that

$$\forall \tau \in H_o^1(I): \quad B_K(\phi_i^o, \tau) = -B_K(N_i, \tau) \tag{3.46}$$

holds for $i = 1, 2$, resp. (cf. [SB, pp. 16/17]).

The real bilinear form

$$B_K(u, v) = (u', v')_I - K^2(u, v)_I$$

is symmetric and coercive (cf. Lemma 2.2). We can find uniquely defined 'one element solutions' N_1^p, N_2^p by solving: Find $\phi_i^p \in V_h = S_o^p(I)$ s.t.

$$\forall \sigma \in V_h: \quad B_K(\phi_i^p, \sigma) = -B_K(N_i, \sigma) \tag{3.47}$$

holds.

We now show eqs (3.41, 3.42). Writing (with the usual summation convention for $j = 3, \ldots, p+1$)

$$\begin{aligned} N_1^p(\xi) &= N_1(\xi) + a_1^j N_j(\xi) \\ N_2^p(\xi) &= N_2(\xi) + a_2^j N_j(\xi) \end{aligned}$$

we find from eq (3.31) the vectors $\{a_1\}, \{a_2\}$ to be

$$\{a_1\} = -[B_{22}]^{-1}[B_{21}]\left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\}; \quad \{a_2\} = -[B_{22}]^{-1}[B_{21}]\left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\}. \tag{3.48}$$

It is now easy to see that

$$[CB] = \left[\begin{array}{cc} B_K(N_1^p, N_1^p) & B_K(N_1^p, N_2^p) \\ B_K(N_2^p, N_1^p) & B_K(N_2^p, N_2^p) \end{array} \right]. \tag{3.49}$$

Indeed,

$$
\begin{aligned}
B_K(N_1^p, N_1^p) &= B_K(N_1, N_1) + B_K(N_1, a_1^m N_m) + B_K(a_1^l N_l, N_1) + B_K(a_1^l N_l, a_1^m N_m) \\
&= B_K(N_1, N_1) + 2\{B_K(N_1, N_m)\}\{a_1\} + \{a_1\}^T[\{B_K(N_l, N_m)\}]\{a_1\} \\
&= B_K(N_1, N_1) - 2\,(\left\{\begin{array}{c} 1 \\ 0 \end{array}\right\}^T [B_{12}])\,([B_{22}]^{-1}[B_{21}]\left\{\begin{array}{c} 1 \\ 0 \end{array}\right\}) \\
&\quad + (\left\{\begin{array}{c} 1 \\ 0 \end{array}\right\}^T [B_{21}]^T [B_{22}]^{-T})\;[B_{22}]\;([B_{22}]^{-1}[B_{21}]\left\{\begin{array}{c} 1 \\ 0 \end{array}\right\}) \\
&= B_{11}[1,1] - \left\{\begin{array}{c} 1 \\ 0 \end{array}\right\}^T [B_{12}]\,[B_{22}]^{-1}\,[B_{21}]\left\{\begin{array}{c} 1 \\ 0 \end{array}\right\} \\
&= CB[1,1]
\end{aligned}
$$

and so forth for $CB[1,2], CB[2,1]$ and $CB[2,2]$. In the equation chain above, we have repeatedly used both the symmetry of the form B_K and the local stiffness matrix.

This validates eq (3.49). Transforming back to global coordinates and assembling, we obtain eqs (3.41, 3.42). $\triangleleft$

In the proof of Theorem 3.2. we will also use the following result.

Lemma 3.1 *Let t_1, t_2 and N_1^p, N_2^p be exact and finite element solutions to eq (3.43) with Dirichlet data (3.44) or (3.45), resp.*

Then for $i = 1,2; j = 1,2$

$$B_K(t_i - N_i^p, t_j - N_j^p) = B_K(N_i^p, N_j^p) - B_K(t_i, t_j) \tag{3.50}$$

holds.

Proof :

Let us specify in eq (3.46)

$$i = 1, \tau = \phi_1^o : \quad B_K(N_1, \phi_1^o) = -B_K(\phi_1^o, \phi_1^o) \tag{3.51}$$
$$i = 2, \tau = \phi_1^o : \quad B_K(N_2, \phi_1^o) = -B_K(\phi_2^o, \phi_1^o) \tag{3.52}$$
$$i = 1, \tau = \phi_2^o : \quad B_K(N_1, \phi_2^o) = -B_K(\phi_1^o, \phi_2^o) \tag{3.53}$$

and, similarly, in eq (3.47)

$$i = 1, \sigma = \phi_1^p : \quad B_K(N_1, \phi_1^p) = -B_K(\phi_1^p, \phi_1^p) \tag{3.54}$$
$$i = 2, \sigma = \phi_1^p : \quad B_K(N_2, \phi_1^p) = -B_K(\phi_2^p, \phi_1^p) \tag{3.55}$$
$$i = 1, \sigma = \phi_2^p : \quad B_K(N_1, \phi_2^p) = -B_K(\phi_1^p, \phi_2^p). \tag{3.56}$$

Furthermore, since $S_o^p \subset H_o^1$,

$$i = 1, \tau = \phi_1^p : \quad B_K(N_1, \phi_1^p) = -B_K(\phi_1^o, \phi_1^p) \tag{3.57}$$
$$i = 2, \tau = \phi_1^p : \quad B_K(N_2, \phi_1^p) = -B_K(\phi_2^o, \phi_1^p) \tag{3.58}$$

which, together with eqs (3.54) and (3.55) shows

$$B_K(\phi_1^o, \phi_1^p) = B_K(\phi_1^p, \phi_1^p) \tag{3.59}$$
$$B_K(\phi_2^o, \phi_1^p) = B_K(\phi_2^p, \phi_1^p). \tag{3.60}$$

We now can check the statement of the lemma by direct computation. For $i = j = 1$ the l.h.s. is

$$\begin{aligned} B_K(t_1 - N_1^p, t_1 - N_1^p) &= B_K(N_1 + \phi_1^o - N_1 - \phi_1^p, N_1 + \phi_1^o - N_1 - \phi_1^p) \\ &= B_K(\phi_1^o, \phi_1^o) - 2B_K(\phi_1^o, \phi_1^p) + B_K(\phi_1^p, \phi_1^p) \\ &= B_K(\phi_1^o, \phi_1^o) - B_K(\phi_1^p, \phi_1^p) \end{aligned}$$

by eq (3.59). The r.h.s. is

$$\begin{aligned} B_K(N_1^p, N_1^p) - B_K(t_1, t_1) &= B_K(N_1 + \phi_1^p, N_1 + \phi_1^p) - B_K(N_1 + \phi_1^o, N_1 + \phi_1^o) \\ &= B_K(N_1, N_1) + 2B_K(N_1, \phi_1^p) + B_K(\phi_1^p, \phi_1^p) \\ &\quad - B_K(N_1, N_1) - 2B_K(N_1, \phi_1^o) - B_K(\phi_1^o, \phi_1^o) \\ &= B_K(\phi_1^o, \phi_1^o) - B_K(\phi_1^p, \phi_1^p) \end{aligned}$$

where we have used eqs (3.54) and (3.51).

This validates the statement for $i = j = 1$. The computation of the remaining cases $i = 1, j = 2$ and $i = 2, j = 1, 2$ is entirely analogous. $\triangleleft$

We are ready to give the proof of Theorem 3.2.

Proof : (Theorem 3.2.) We will show that, neglecting higher order terms of (kh),

$$|\cos(k'h) - \cos(kh)| \leq Ck^2h^2 \frac{C_a^2(p)}{4} \left(\frac{hk}{2p}\right)^{2p} \tag{3.61}$$

holds with an independent constant C. Assuming this relation for the moment we continue with ($C_1(p) := CC_a^2(p)/4$)

$$\left|2\sin\frac{k'+k}{2}h\sin\frac{k'-k}{2}h\right| = |\cos(kh)-\cos(k'h)| \le k^2h^2C_1(p)\left(\frac{hk}{2p}\right)^{2p}.$$

Let $k'h - kh = \varepsilon$, then

$$|2\sin\frac{\varepsilon}{2}| \le \left|\frac{k^2h^2C_1(p)\left(\frac{hk}{2p}\right)^{2p}}{\sin(kh+\frac{\varepsilon}{2})}\right| \le \frac{k^2h^2C_1(p)\left(\frac{hk}{2p}\right)^{2p}}{kh}.$$

Since by assumption $(hk/2p)^{2p} \ll 1$ we see that ε is small and we may neglect higher order terms in the Taylor expansion of $2\sin\frac{\varepsilon}{2}$. Thus we obtain

$$\varepsilon \le hkC_1(p)\left(\frac{hk}{2p}\right)^{2p}$$

and replacing $\varepsilon = k'h - kh$ we conclude the statement of the theorem.

We now have to prove eq (3.61) which we write in the form

$$\left|\frac{S_p}{T_p}-\frac{S_o}{T_o}\right| \le k^2h^2C_1(p)\left(\frac{hk}{2p}\right)^{2p}. \tag{3.62}$$

Consider any internal element Δ being mapped on the master element $I = (-1,1)$. On I, the exact homogeneous solution of the VP (1.4) is represented by

$$u_{ex}(\xi) = u_{ex}^1 t_1(\xi) + u_{ex}^2 t_2(\xi)$$

where

$$\begin{aligned} t_1(\xi) &= -\frac{\sin K\xi}{2\sin K} + \frac{\cos K\xi}{2\cos K} \\ t_2(\xi) &= \frac{\sin K\xi}{2\sin K} + \frac{\cos K\xi}{2\cos K}. \end{aligned}$$

are solutions of the BVP's (3.43, 3.44) or (3.43, 3.45), resp., with $K = kh/2$.

The finite element solution is written on I as

$$u_{fe}(\xi) = u_{fe}^1 N_1^p(\xi) + u_{fe}^2 N_2^p(\xi)$$

where N_1^p, N_2^p are approximate solutions to the BVP's (3.43, 3.44) or (3.43, 3.45), as discussed in the preliminaries.

By continuity of B_K,

$$|B_K(t_l - N_l, t_m - N_m)| \le (1+K^2)|t_l - N_l|_1 |t_m - N_m|_1,$$

and applying now eq (3.9) we have

$$|B_K(t_l - N_l, t_m - N_m)| \le (1+K^2)\frac{C_a(p)^2}{p^{2p}} |t_l|_{p+1} |t_m|_{p+1} \tag{3.63}$$

for $l, m = 1, 2$.

By direct computation,

$$|t_l|_{p+1}^2 = \begin{cases} K^{2p+2}\|t_l\|^2 & \text{if} \quad p \quad \text{is odd} \\ K^{2p}|t_l|_1^2 & \text{if} \quad p \quad \text{is even} \end{cases} \tag{3.64}$$

with

$$\|t_l\|^2 = \frac{2}{3} + \mathcal{O}(K^2); \quad |t_l|_1^2 = \frac{1}{2} + \mathcal{O}(K^4).$$

Here and in the following, $\mathcal{O}(K^2)$ means an expression of the form $C_1K^2 + C_2K^4 + \ldots$ with constants C_i not depending on h, k and p.

Also by direct computation,

$$B_K(t_1, t_1) = B_K(t_2, t_2) = \frac{1}{2} + \mathcal{O}(K^2) \tag{3.65}$$

and

$$B_K(t_1, t_2) = B_K(t_2, t_1) = -\frac{1}{2} + \mathcal{O}(K^2). \tag{3.66}$$

Finally, we recall from Lemma 3.1 that

$$B_K(t_l - N_l, t_m - N_m) = B_K(N_l, N_m) - B_K(t_l, t_m). \tag{3.67}$$

holds for $l, m = 1, 2$.

Returning to the proof of eq (3.62), we have

$$\begin{aligned} |S_pT_o - S_oT_p| &= |B_K(t_1,t_1)B_K(N_1,N_2) - B_K(t_1,t_2)B_K(N_1,N_1)| \\ &= |B_K(t_1,t_1)\left(B_K(t_1-N_1,t_2-N_2) - B_K(t_1,t_2)\right) - \\ &\quad\ B_K(t_1,t_2)\left(B_K(t_1-N_1,t_1-N_1) - B_K(t_1,t_1)\right)| \\ &= |B_K(t_1,t_1)B_K(t_1-N_1,t_2-N_2) \\ &\quad\ -B_K(t_1,t_2)B_K(t_1-N_1,t_1-N_1)| \end{aligned} \tag{3.68}$$

Here, we applied eq (3.67) to expand the expression on the r.h.s. Thus

$$\begin{aligned} |S_pT_o - S_oT_p| &\leq |B_K(t_1,t_1)||B_K(t_1-N_1,t_2-N_2)| + \\ &\quad |B_K(t_1,t_2)||B_K(t_1-N_1,t_1-N_1)| \end{aligned}$$

and applying eqs (3.63, 3.65, 3.66) leads to

$$|S_pT_o - S_oT_p| \leq \left(\frac{1}{2} + \mathcal{O}(K^2)\right)\left((1+K^2)\frac{C_a(p)^2}{p^{2p}}\left(|t_1|_{p+1}|t_2|_{p+1} + |t_1|^2_{p+1}\right)\right).$$

For odd p we then have directly by (3.64)

$$\begin{aligned} |S_pT_o - S_oT_p| &\leq \left(\frac{1}{2} + \mathcal{O}(K^2)\right)\left((1+K^2)\frac{C_a(p)^2}{p^{2p}}K^{2p+2}\left(\frac{4}{3} + \mathcal{O}(K^2)\right)\right) \\ &\leq \frac{C_a(p)^2}{p^{2p}}K^{2p+2} \end{aligned} \tag{3.69}$$

where we neglected terms of order $\mathcal{O}(K^2)$. Then also (3.62), and hence the statement, holds since

$$\left|\frac{S_p}{T_p} - \frac{S_o}{T_o}\right| = \frac{|S_pT_o - S_oT_p|}{|T_oT_p|}$$

and it can easily be seen that

$$|T_oT_p| = |B_K(t_1,t_2)||B_K(t1 - N1, t_2 - N_2) + B_K(t_1,t_2)|$$

is bounded from below by a constant independently from K, p.

If p is even then we insert eqs (3.65, 3.66) into eq (3.68) to get

$$\begin{aligned} |S_pT_o - S_oT_p| &\leq \frac{1}{2}|B_K(t_1-N_1, t_1+t_2-(N_1+N_2))| + \\ &\quad \mathcal{O}(K^2)\ \left(|B_K(t_1-N_1,t_2-N_2)| + |B_K(t_1-N_1,t_1-N_1)|\right) \end{aligned}$$

For the first term in this equation we have

$$\begin{aligned} |B_K(t_1-N_1, t_1+t_2-(N_1+N_2))| &\leq \frac{C_a(p)^2}{p^{2p}}|t_1|_{p+1}\ |(t_1+t_2)|_{p+1} \\ &\leq \frac{C_a(p)^2}{p^{2p}}K^{2p}|t_1|_1\left|\frac{\cos K\xi}{\cos K}\right|_1 \\ &\leq CC_a(p)^2\frac{K^{2p+2}}{p^{2p}} \end{aligned}$$

with C not depending on K, p. Again, terms of order $\mathcal{O}(K^2)$ have been neglected. For the second term, a similar estimate follows directly from eqs (3.63,3.64). Thus the estimate (3.69) holds also for even p, and the statement follows by similar argument. $\triangleleft$

3.4 Discrete stability

In Part I, when investigating the h-version with $p = 1$, we proved the stability estimate

$$\|u'_{fe}\| \leq C\|f\|$$

and showed that the inf-sup-constant on the discrete subspace is $\gamma_h = Ck^{-1}$. A standard corollary then yields the stability estimate

$$\|u'_{fe}\| \leq Ck\|f^{-1}\|.$$

We will now show that both results carry over to higher p.

Further we will define in this subsection a specific data subspace that we will encounter in the error analysis. In this subspace we will prove stability w.r. to higher integrals of the data - the discrete analogon to Theorem 2.2.

Let hence $u_{fe} \in S_h^p$ be the finite element solution to the VP (1.4) for data $f \in L^2(\Omega)$. We write

$$u_{fe} = u_h + u_p$$

where u_h is based on the nodal shape functions and u_p on the internal ones. From the definition of the shape functions N_j and orthogonality property of the Legendre polynomials it follows that

$$\|u'_{fe}\|^2 = \|u'_h\|^2 + \|u'_p\|^2. \tag{3.70}$$

Furthermore, u_p satisfies on each element Δ eq (3.18). Transforming $\Delta \to I^+ = (0,1)$, we get

$$\forall w \in S_o^p(\Delta): \quad B_K(u_p, w) = h^2(f, w) + K^2(u_h, w). \tag{3.71}$$

We now prove a first stability lemma on $\|u'_p\|$ for data $f \in L^2(\Omega)$.

Lemma 3.2 *Let, for u_{fe} be the finite element solution to the VP (1.4) with data $f \in L^2(\Omega)$. Assume that $hk \leq \alpha < \pi$.*

Then

$$\|u'_{fe}\| \leq C\|f\| \tag{3.72}$$

holds with a constant C independent of h,k and p.

Proof : Let $u_{fe} = u_h + u_p$ as defined above. We know (cf. part I, Lemma 3) by straight estimation of the discrete Green's function representation that $\|u_h'\| \le C_1 \|f\|$, where C_1 does not depend on h, k and p. Applying eq (2.5) from Lemma 2.2 and Remark 2 to eq (3.71), we have

$$\|u_p'\| \le D\left(h^2\|f\| + K^2\|u_h\|\right)$$

where $D = \pi/(\pi^2 - \alpha^2)$. Back-transform to Δ then yields

$$\|u_p'\|_\Delta \le Dh\left(\|f\|_\Delta + k^2\|u_h\|_\Delta\right).$$

Summing up and applying Schwarz inequality, we get

$$\|u_p'\| \le Dh\left(\|f\| + k^2\|u_h\|\right) \le \|f\| h\left(D + C_2 k\right)$$

where we applied $k\|u_h\| \le C_2\|f\|$ with C_2 not depending on h, k, p. Thus

$$\|u_{fe}'\| \le (C_1 + Dh + C_2 kh)\ \|f\|,$$

and the statement is readily obtained by neglecting Dh and setting $C = C_1 + C_2\alpha$. The proof is completed. $\lhd$

We now prove the inf-sup-condition for $V_h = S_h^p(\Omega)$.

Lemma 3.3 (Discrete inf-sup-condition): *Let $V_h = S_h^p(\Omega)$ and let $B : V_h \times V_h \to \mathbb{C}$ be the sesquilinear form defined by eq (1.4).*
Then, if h is such that $hk \le \alpha < \pi$,

$$\inf_{u\in V_h} \sup_{v\in V_h} \frac{|B(u,v)|}{|u|_1 |v|_1} \ge \frac{C}{k} \tag{3.73}$$

where C does not depend on h, k and p.

Proof : The argument is similar to the case $p = 1$. For arbitrarily fixed $u \in V_h$ set $v := u + z$ where $z \in V_h$ is solution to the VP

$$\forall w \in V_h : \quad B(w,z) = k^2(w,u).$$

Let $z := z_h + z_p$ as above. In part I it was shown that

$$\|z_h'\| \le C_2 k\left(\frac{k}{k'}\right)\|u'\|.$$

The ratio k/k' is bounded for $kh < \pi$ and $p \geq 2$ by Theorem 3.2. Further, z_p solves locally, i.e. for all $w \in S_o^p(\Delta)$,

$$B_\Delta(z_p, w) = k^2(u + z_h, w)_\Delta.$$

Applying Lemma 2.2 and Remark 2,

$$\|z_p'\| \leq C_1 k(\|u\| + \|z_h\|).$$

Again by discrete Green's function representation, we can show that $\|z_h\| \leq C_2\|u'\|$, with C_2 not depending on h, k and p. Applying a Poincaré inequality for $\|u\|$, we conclude

$$\|z'\| \leq C_3 k\|u'\|$$

and the statement is concluded, using the particular choice of z (cf. part I). $\lhd$

It follows by standard theory that $|u|_1 \leq Ck\|f^{(-1)}\|$.
We collect both stability estimates in the following proposition.

Theorem 3.3 (Stability of the FE-solution I): *Let $f \in L^2(\Omega)$ and let $u_{fe} \in V_h = S_h^p(\Omega)$ be the finite element solution to the VP (1.4).*
Then, if h is such that $hk \leq \alpha \leq \pi$, the stability estimates

$$|u_{fe}|_1 \leq C_1\|f\| \tag{3.74}$$

and

$$|u_{fe}|_1 \leq C_2 k\|f^{(-1)}\| \tag{3.75}$$

hold for C_1, C_2 not depending on h, k and p.

Next we formulate a dual stability property that we will use to prove a preasymptotic error estimate for the p-degree finite element solution. To this end, we define a specific data subspace.

Definition 3.1 *For integer $l \geq 0$ we define a subspace $F_o^l(\Omega) \subseteq L^2(\Omega)$ by*

$$F_o^l(\Omega) = \left\{ f \in L^2(\Omega) \;\middle|\; f^{(-i)}|_{X_h} = 0 \quad for \quad i = 1 \dots l \right\}$$

with $F_o^o(\Omega) := L^2(\Omega)$.

Observe that for this space we can show, by adding up local Poincaré inequalities,

$$\|f^{(-i)}\| \le h\|f^{(-i+1)}\|. \tag{3.76}$$

In particular, for $f \in F_o^1$ and $hk \le \alpha$, inequality (3.74) directly follows from (3.75).

For the discrete data obtained from $f \in F_o^l(\Omega)$ by the finite element procedure, the following proposition is true.

Lemma 3.4 *Consider the VP (1.4) on $V_h = S_h^p(\Omega)$ with data $f \in F_o^{p-1}(\Omega)$. Let Δ_j be an arbitrary finite element and let $\{\tilde{r}_{j-1}, \tilde{r}_j\}^T$ be the condensed right hand side vector given by eq (3.26). Assume further that the stepwidth h is sufficiently small so that $hk \le \alpha < \pi$.*

Then

$$|\tilde{r}_j| \le C_d(p,m)h^{1/2}k^m\|f^{(-m)}\|_{\Delta_i} \tag{3.77}$$

holds for even $m = 0, 2, \ldots \le p-1$ with

$$C_d(p,0) = 1$$

and

$$C_d(p,m) = C_1 + C_2\alpha^{p-m}\, 2^{(m-1)/2}\frac{(p+1)!(p+1)}{((p-m+1)!)^2}, \quad m \ge 2$$

where C_1, C_2 do not depend on h, k and p.

Proof : Let first $m = 0$. We omit in the notation the element number j and renumber formally $\{\tilde{r}_{j-1}, \tilde{r}_j\} \to \{\tilde{r}_1, \tilde{r}_2\}$. Eq (3.26) then reads

$$\left\{ \begin{array}{c} \tilde{r}_1 \\ \tilde{r}_2 \end{array} \right\} = \left\{ \begin{array}{c} r_1 \\ r_2 \end{array} \right\} - [B_{12}]\,[B_{22}]^{-1} \left\{ \begin{array}{c} r_3 \\ \vdots \\ r_{p+1} \end{array} \right\}.$$

It is straightforward to show that $r_1 = (f, N_1)_\Delta$ and $r_2 = (f, N_2)_\Delta$ satisfy $|r_j| \le C_1 h^{1/2}\|f\|_\Delta$, $j = 1, 2$.

Define $\{y\} := [B_{22}]^{-1}\{r_3, \ldots, r_{p+1}\}^T$. This vector is the discrete solution to the VP

$$\forall w \in S_o^p(\Delta): \quad B_\Delta(y, w) = (f, w)_\Delta$$

and by Lemma 2.2 and Remark 2,

$$\|y'\| \le h\frac{\pi}{\pi^2 - \alpha^2}\|f\|_\Delta.$$

By simple computation, $\|y'\|_\Delta^2 \geq \frac{1}{h}|\{y\}|^2$, hence

$$|\{y\}| \leq C_1 h^{3/2}\|f\|_\Delta,$$

where $C_1 = \pi/(\pi^2 - \alpha^2)$ does not depend on h, k and p.

Consider now the term $\{z\} := [B_{12}][B_{22}]^{-1}\{r_3, \ldots, r_{p+1}\}^T = [B_{12}]\{y\}$. The coefficients of the matrix $[B_{12}]$ are $b_{ij} = B_\Delta(N_i, N_j)$ with $i = 1,2$ and $j = 3\ldots p+1$. For these i, j we can, using integration by parts, that $(N_i', N_j')_\Delta = 0$ and hence $b_{ij} = -k^2(N_i, N_j)_\Delta$. Obviously, the euclidian norm of the rows $\{b_i\}, i = 1,2$ can be bounded as

$$|\{b_i\}| \leq C_2 k^2 h.$$

The constant C_2 does not depend neither on h, k nor on p since the bandwidth of the local mass matrix does not increase with p for $p \geq 3$ (cf. [SB, p.46]).

Thus, with the previous estimate of $\|\{y\}\|$, for $i = 1,2$,

$$\begin{aligned} |z_i| &= |b_{ij}y_j| \leq |\{b_i\}|_\Delta |\{y\}|_\Delta \\ &\leq C_1 C_2 k^2 h^{5/2}\|f\|_\Delta \\ &\leq C_3 h^{1/2}\|f\|_\Delta \end{aligned} \tag{3.78}$$

with $C_3 = C_1 C_2 \alpha^2$. Together with the observation for r_1, r_2, the last estimate proves the statement for $m = 0$.

Let next $m \geq 2$ and assume for convenience that Δ has been mapped to $I^+ = (0,1)$. Then, for $j = 1,2$,

$$\tilde{r}_j = h\int_0^1 f(\theta)\varphi_j(\theta)d\theta \tag{3.79}$$

where $\varphi_1, \varphi_2 \in S^p_{(o}(I^+)$ or $S^p_{o)}(I^+)$, resp., are the 'one-element' solutions to the homogeneous Helmholtz equation

$$u'' + k^2h^2u = 0 \tag{3.80}$$

with the boundary conditions

$$u(0) = 1, \;\; u(1) = 0 \tag{3.81}$$

or

$$u(0) = 0, \;\; u(1) = 1, \tag{3.82}$$

resp. The exact solutions t_1 and t_2 of these BVP are

$$t_1(\theta) = \cos kh\theta - \cot kh \sin kh\theta \tag{3.83}$$

$$t_2(\theta) = \sin kh\theta / \sin kh. \tag{3.84}$$

Integration by parts in eq (3.79) leads to

$$|\tilde{r}_j| = h\,|\int_0^1 f^{(-m)}(\theta)\varphi^{(m)}(\theta)d\theta|;$$

no boundary terms occur for $m \le p-1$ since $f \in F_o^{p-1}(\Omega)$. Consequently we have for $j = 1, 2$

$$|\tilde{r}_j| \le h\,\|f^{(-m)}\|_{I^+}\|\varphi_j^{(m)}\|_{I^+}.$$

For the estimation of $\|\varphi_j^{(m)}\|_{I^+}$, we define $\chi_j \in S^p_{(o}(I^+)$ or $S^p_{o)}(I^+)$, resp., by

$$\chi_1(\theta) := \tau_1(\theta) - \theta\tau_1(1)$$

and

$$\chi_2(\theta) := \tau_2(\theta) + \theta(1 - \tau_2(1))$$

where $\tau_1, \tau_2 \in S^p(I^+)$ are the Taylor polynomials of order p in $\theta_o = 0$ for $t_1(\theta)$ and $t_2(\theta)$, resp. Now trivially

$$\|\varphi_j^{(m)}\|_{I^+} \le \|\varphi_j^{(m)} - \chi_j^{(m)}\|_{I^+} + \|\chi_j^{(m)}\|_{I^+}.$$

As to the second member on the r.h.s., it can be shown by direct computation that for even $m \ge 2$

$$\|\chi_j^{(m)}\|_{I^+} \le C_1(hk)^m. \tag{3.85}$$

holds with a constant C_1 not depending on h, k and p. Turn to the estimation of $\|\varphi_j^{(m)} - \chi_j^{(m)}\|_{I^+}$. For $m = 1$, we have

$$|\varphi_j - \chi_j|_1 \le |t_j - \varphi_j|_1 + |t_j - \chi_j|_1 \le C_2|t_j - \chi_j|_1 \tag{3.86}$$

by Céa's lemma. [4] By construction of χ_j,

$$|t_j - \chi_j|_1 = |t_j - \tau_j| + \mathcal{O}\left(\frac{(hk)^p}{p!}\right) \le C_3\left(\frac{(hk)^p}{p!}\right), \tag{3.87}$$

where C_3 does not depend on h, k and p.

For $m \ge 2$, we use repeatedly the inverse inequality (3.2) to relate

$$\|\varphi_j^{(m)} - \chi_j^{(m)}\|_{I^+} \le \left(\frac{(p+1)!}{(p-m+1)!}\right)^2 2^{(m-1)/2}|\varphi_j - \chi_j|_1.$$

[4] To be precise, we write χ, φ as sums of linear and bubble functions and apply the statement of Céa's lemma on an appropriate subspace $V_h \subset H_o(I^+)$.

Inserting this estimate into (3.86) vs. (3.87), we conclude that

$$\|\varphi_j^{(m)} - \chi_j^{(m)}\|_{I^+} \leq C_4 \frac{((p+1)!)^2\, 2^{(m-1)/2}}{((p-m+1)!)^2 p!}(hk)^p$$

where $C_4 = C_2C_3$ does not depend on h, k and p.

Thus

$$\|\varphi_j^{(m)}\|_{I^+} \leq C_1(hk)^m + C_4 \frac{(p+1)!(p+1)}{((p-m+1)!)^2}\, 2^{(m-1)/2}(hk)^p$$

and, finally

$$|\tilde{r}_j| \leq h(hk)^m \left(C_1 + C_4 \frac{(p+1)!(p+1)}{((p-m+1)!)^2}\, 2^{(m-1)/2}(hk)^{p-m} \right).$$

The statement now follows by back-transform $I^+ \to \Delta$. The proof is completed. $\triangleleft$

Remark 9: For odd m this statement holds true with the additional assumption that hk is bounded from below, i.e. $0 < \beta \leq hk$.

Let us show this for $m = 1$ and $j = 2$. We have

$$\|\chi_2'\| \leq \|\tau_2'\| + \|1 - \tau_2(1)\|$$

By construction, the second member is $\mathcal{O}\left(\frac{(hk)^p}{p!}\right)$. Also $\tau_2'(\theta) = t_2'(\theta) + \mathcal{O}\left(\frac{(hk)^p}{p!}\right)$. Hence, neglecting terms of higher order (note that by assumption $p > m$),

$$\begin{aligned} \|\chi_2'\| &\leq \|t_2'\| = \frac{kh}{\sin kh}\|\cos kh\theta\|_{I^+} \\ &\leq C_1\left(1 + \mathcal{O}\left(\frac{(kh)^2}{2}\right)\right). \end{aligned}$$

Furthermore, $|\varphi_2 - \chi_2|_1 = \mathcal{O}\left(\frac{(hk)^p}{p!}\right)$. Hence $\|\varphi_2'\|_{I^+} \leq C_2$ and

$$\begin{aligned} |\tilde{r}_2| &\leq C_2\, h\, \|f^{(-1)}\|_{I^+} \\ &\leq C_2 h^{-1/2}\|f^{(-1)}\|_\Delta \end{aligned}$$

after back-transform $I^+ \to \Delta$. Multiplying and dividing now by $kh^{1/2}$, we get

$$\begin{aligned} |\tilde{r}_2| &\leq C_2 \frac{h^{1/2}k}{hk}\|f^{(-1)}\|_{I^+} \\ &\leq Ch^{1/2}k\|f^{(-1)}\|_\Delta \end{aligned}$$

with $C = C_2\beta^{-1}$. This shows our case; the argument for $j = 1$ or higher m is similar.

Corollary 3.1 *For the norm of the discrete right hand side R_h, the estimate*

$$\|R_h\|_h \leq C_d(p,m)\, h\, k^m \|f^{(-m)}\| \tag{3.88}$$

holds for $m = 0 \ldots \leq p-1$.

Proof : By definition (cf. part I),

$$\|R_h\|_h^2 = h \sum_{i=1}^{n} |R_i|^2$$

where the coefficients R_i are given by eq (3.30). By Lemma 3.4,

$$\|R_h\|_h \;\leq\; C_d(p,m) \left(h \sum_{i=1}^{n} h k^{2m} \|f^{(-m)}\|_{\Delta_i}^2 \right)^{1/2}$$

and the statement readily follows. ◁

We now prove a proposition on dual stability if the data $f \in F_o^m(\Omega)$.

Theorem 3.4 (Stability of the FE-solution II): *Let $u_{fe} \in V_h = S_h^p(\Omega)$ be the FE-solution to the VP (1.4) with data $f \in F_o^m(\Omega)$, where m is even, $m \leq p-1$. Assume that $0 < kh \leq \alpha < \pi$.*

Then

$$|u_{fe}|_1 \leq C_d(p,m) k^m \|f^{(-m)}\| + C_1 \|f^{(-1)}\|, \tag{3.89}$$

where C_d is the discrete dual stability constant (cf. Lemma 3.4) and C_1 does not depend on h,k and p.

Proof : In light of Theorem 3.3. we only need to prove the statement for $m \geq 2$. As before, we write $u_{fe} = u_h + u_p$. Then

$$\|u'_{fe}\| \leq \|u'_h\| + C_1(\|f^{(-1)}\| + k^2 \|u_h^{(-1)}\|)$$

by Lemma 3.2. It can be shown, using the Green's function representation of u_h that

$$\|u'_h\| + C_1 k^2 \|u_h^{(-1)}\| \leq \frac{C_2}{h} \|R\|_h.$$

where C_2 does not depend on h, k and p. On the other hand,

$$\|R\| \leq C_d(p,m) h k^m \|f^{(-m)}\|,$$

from Lemma 3.4, and the statement follows. The proof is completed ◁

Remark 10: The stability theorem holds for odd m with the additional assumption that $0 < \beta \leq hk$ - cf. the previous remark.

3.5 Error estimates for the finite element solution

Throughout this subsection, we denote by l the regularity of the exact solution u, i.e. we assume that $u \in V^l := H^{l+1}(\Omega) \cap V$ with $V = H^1_{(o}(\Omega)$.

In part I we proved that the finite element solution is asymptotically quasioptimal. The same result holds for higher approximation. However, the range of asymptotic behaviour, as taken w.r. to the meshsize h^{-1}, grows with p.

Theorem 3.5 (Asymptotic estimate): *Let $l \geq 1, p \geq 1$. Let further $u \in V^l$ and $u_{fe} \in V_h = S_h^p(\Omega)$ be the exact and finite element solutions to the VP (1.4), resp.*

Then, if k^2h/p is sufficiently small, the quasioptimal estimate

$$|u - u_{fe}|_1 \leq C \inf_{v \in V_h} |u - v|_1 \tag{3.90}$$

holds with

$$C = \left(\frac{4 + \left(\frac{hk}{2p}\right)^2}{\frac{1}{2} - 6k^2 \left(\frac{hk}{p}\right)^2 \left(1 + \sqrt{\frac{3}{2}} \left(\frac{hk}{p}\right)^2\right)} \right)^{1/2} . \tag{3.91}$$

provided the denominator of C is positive.

Proof : Denote $e := u - u_{fe}$ and let z be the solution to the VP

$$\forall w \in V : \quad B(w, z) = (w, e). \tag{3.92}$$

This problem has a unique solution $z \in H^3(\Omega)$. We can show (part I, Theorem 3) that

$$\|e\|^2 \leq 2\left(|z - w|_1 |e|_1 + k^2 \|z - w\| \|e\|\right)$$

holds for all $w \in V$. In particular, for $w = s$, where s is the approximation of z as constructed in Theorem 3.1., we have

$$\|e\|^2 \leq 2\left(\frac{h}{2p} |z|_2 |e|_1 + k^2 \left(\frac{h}{2p}\right)^3 |z|_3 \|e\|\right).$$

We now apply the stability estimates (part I, Lemma 1, and eq (2.24), Theorem 2.1) to get

$$\begin{aligned} |z|_2 &\leq (1 + k)\|e\| \\ |z|_3 &\leq (1 + 4k)\|e\|_1 \leq \sqrt{\frac{3}{2}}(1 + 4k)|e|_1 \end{aligned}$$

by Poincaré inequality since $e \in H_{(o}(\Omega)$.

Hence

$$\|e\|^2 \leq 2\left(\frac{h}{2p}(1+k) + k^2(1+4k)\left(\frac{h}{2p}\right)^3\sqrt{\frac{3}{2}}\right)\|e\|\,|e|_1;$$

dividing now by $\|e\|$ and neglecting terms where h is of order higher than k, we arrive at the intermediary result

$$\|e\| \leq \frac{kh}{p}(1+\sqrt{\frac{3}{2}}\left(\frac{kh}{p}\right)^2)\,|e|_1. \tag{3.93}$$

In the next step we use B-orthogonality of e to show (cf. again part I, Theorem 3) that

$$\frac{1}{2}|e|_1^2 \leq 6k^2\|e\|^2 + 4|u-v|_1^2 + k^2\|u-v\|^2$$

holds for all $v \in V_h$. We choose $v = s$ from Theorem 3.2, then

$$\|u-s\| \leq \frac{h}{2p}|u-s|_1$$

and applying now eq (3.93) we obtain

$$\frac{1}{2}|e|_1^2 - 6k^2\left(\frac{kh}{p}\right)^2(1+\sqrt{\frac{3}{2}}\left(\frac{kh}{p}\right)^2)^2|e|_1^2 \leq (4+\frac{k^2h^2}{4p^2})|u-s|_1^2$$

hence

$$|e|_1 \leq C|u-s|_1$$

where

$$C = \left(\frac{4+\left(\frac{hk}{2p}\right)^2}{\frac{1}{2} - 6k^2\left(\frac{hk}{p}\right)^2(1+\sqrt{\frac{3}{2}}\left(\frac{hk}{p}\right)^2)}\right)^{1/2}.$$

This completes the proof. $\triangleleft$

Remark 11: For piecewise linear approximation we proved (part I, Corollary 2)

$$|u-u_{fe}| \leq C_1 \inf_{v\in V_h}|u-v|_1$$

with

$$C_1 = \frac{2\left(1+\left(\frac{hk}{2\pi}\right)^2\right)^{\frac{1}{2}}}{\left(\frac{1}{2} - 6C_I^2k^2h^2(1+k)^2\right)^{\frac{1}{2}}}$$

where

$$C_2 = \frac{2}{(1 - 2(1+k)\frac{k^2h^2}{\pi^2})\,\pi}.$$

Obviously $C_2 \geq 2$, hence $k^4h^2 \leq \frac{1}{12 \cdot 4}$ is necessary for well-definiteness of C_1. A similar computation yields $k^4h^2 \leq \frac{p^2}{12}$ as a necessary condition for well-definiteness of C in the theorem above.

Remark 12: In the form given in Theorem 3.5, the quasioptimal estimate holds independently on the regularity of the solution u. The order of convergence, in terms of hp^{-1}, is obtained by introcucing the approximation property of the subspace V_h from Theorem 3.1. For given p, the maximal order of convergence

$$|u - u_{fe}|_1 \leq C\,(e/2)^{2p}\,(\pi p)^{-1/4}\left(\frac{h}{2p}\right)^p |u|_{p+1}$$

is achieved when the regularity of u is $l \geq p$.

We now proceed to error estimates in the preasymptotic range (i.e. without restrictions on k^2h). Let us first relate the finite element solution to best approximations in $V_h = S_h^p(\Omega)$, as constructed in Theorem 3.1.

Lemma 3.5 *Let, for $p \geq 1$, u and u_{fe} be the exact and finite element solutions to the VP (1.4), resp., and let $s \in V_h$ be a nodally exact quasioptimal approximation to u in the sense of Theorem 3.1.*

Then $z := u_{fe} - s$ is the finite element solution to the VP (1.4) with data $k^2(u-s)$.

Proof : Trivially $z = u_{fe} - u + u - s$, and by B-orthogonality of $u - u_{fe}$ to V_h, $B(z,v) = B(u-s,v)$ holds for all $v \in V_h$. The boundary term in $B(u-s,v)$ vanishes due to $u|_{X_h} = s|_{X_h}$. Using local exactness of the integrals of s we show, repeatedly integrating by parts, that also the term $((u-s)', v')$ vanishes. Thus, for all $v \in V_h$,

$$B(u-s,v) = -k^2(u-s,v)$$

which completes the proof. ◁

It is now straightforward to show a first error estimate.

Theorem 3.6 (Preasymptotic estimate I): *Let, for $1 \leq l \leq p$, $u \in V^l$ and $u_{fe} \in V_h$ be the solution and the finite element solution to the VP (1.4), respectively. Assume that $hk \leq \alpha < \pi$.*

Then for $e := u - u_{fe}$

$$|e|_1 \le C_a(l)\left(1 + C_1 k\left(\frac{kh}{2p}\right)\right)\left(\frac{h}{2p}\right)^l |u|_{l+1} \tag{3.94}$$

holds, where C_1 does not depend on h, k and p, whereas $C_a(l)$ is the approximation constant (Theorem 3.1), being at most of order $\left(\frac{e}{2}\right)^p$.

Proof : Let $z = u_{fe} - u + u - s$ as above. By Theorem 3.3 and the previous lemma, $|z|_1 \le Ck^2\|u - s\|$, hence

$$|e|_1 = |z + u - s|_1 \le Ck^2\|u - s\| + |u - s|_1. \tag{3.95}$$

To complete the proof, we insert the appropriate results from the approximation theorem. ◁

Remark 13: Note that, if $k^2h/2p$ is bounded, the error estimate is equivalent to the asymptotic quasioptimal estimate given in Theorem 3.5. This, again, is an analogy to the h-version with $p = 1$ (part I, section 3.6).

The estimate of the previous lemma can be generalized for $p \ge 2$, employing dual stability estimates for the data $k^2(u - s) \in F_o^{p-1}(\Omega)$.

Theorem 3.7 (Preasymptotic estimate II): *Let $1 \le l \le p$ and $0 \le m \le p$, m even, with $p \ge 2$. Let $u \in V^l$ be the solution to the VP (1.4) with data $f \in H^{(l-1)}(\Omega)$ and let $u_{fe} \in V_h$ be the finite element solution to this problem. Assume further that the stepwidth h is such that $hk \le \alpha < \pi$.*

Then

$$|e|_1 \le C_a(l)\left[1 + C_1\left(\frac{kh}{2p}\right)^2 + kC_d(p,m)C_a(m)\left(\frac{kh}{2p}\right)^{m+1}\right]\left(\frac{h}{2p}\right)^l |u|_{l+1}. \tag{3.96}$$

holds with C_1 not depending on k, h, p.

Proof : Let $s \in V_h$ be a nodally exact, optimal approximation of u in the sense of Theorem 3.1 and define, as before, $z := u_{fe} - u_s$. We know that z solves $B(z, v) = -k^2(u - s, v)$ for all $v \in V_h$. The data of this problem is in the space F_o^{p-1}, hence by Theorem 3.4.

$$|z|_1 \le k^2\left(C_d(p,m)k^m\|(u - s)^{(-m)}\| + C_1\|(u - s)^{(-1)}\|\right)$$

holds for $m \leq p-1$. Inserting the from Theorem 3.1

$$\|(u-s)^{(-m)}\| \leq C_a(l)C_a(m)\left(\frac{h}{2p}\right)^{l+m+1}|u|_{l+1},$$

we conclude the statement. ◁

Remark 14: With the additional assumption $0 < \beta \leq kh$, the statement holds also for odd m - cf. remarks 9,10. This assumption is consistent with the error estimation in the *preasymptotic* range and with computational application where the magnitude of hk is, for medium and high k, bounded from below by practical considerations.

Remark 15: We obtain estimate I from estimate II by setting $m = 0$, hence II generalizes I.

Let us specify the estimate II for a certain type of solutions, namely, those oscillating with frequency k. These solutions are of practical importance in physical applications in wave propagation and wave scattering; they are, among others, produced by Dirac data (point sources).

Thus, having in mind to specify solutions that essentially behave like $\exp(ikx)$, we define:

Definition 3.2 *Let, for $l \geq 1$, $u \in V^l$ be a solution to the VP (1.4). We call u an* oscillating *solution if*

$$|u|_{l+1} \leq Dk^l|u|_1 \tag{3.97}$$

holds with a constant D not depending on k.

With this definition, we directly have the following corollary.

Corollary 3.2 (Error estimate for oscillating solutions): *Let $1 \leq l \leq p$ and $p \geq 2$. Assume that there is given a data $f \in H^{l-1}(\Omega)$ such that the solution $u \in V^l$ to the VP (1.4) is oscillating. Assume further that the stepwidth h is such that $hk \leq \alpha < \pi$. Let $u_{fe} \in V_h = S_h^p(\Omega)$ be the finite element solution to the VP (1.4).*

Then the relative error $|\tilde{e}|_1 := |u-u_{fe}|_1/|u|_1$ is bounded by

$$|\tilde{e}|_1 \leq \left(\frac{hk}{2p}\right)^l\left[C_1 + C_2\left(\frac{kh}{2p}\right)^2\right] + kC_3\left(\frac{kh}{2p}\right)^{l+m+1}. \tag{3.98}$$

where $C_1 = DC_a(l), C_2 = EC_a(l)$ *and* $C_3 = DC_d(p,m)C_a(l)C_a(m)$, *with* D, E *not depending on* h, k *and* p.

Proof : We introduce the definition of oscillatory behaviour into the estimate II, eq (3.96). ◁

Let us consider special cases of the previous corollary.

1. If $l \geq p$ we have with $\theta := \left(\frac{hk}{2p}\right)^p$

   $$|\tilde{e}|_1 \leq \theta(C_1 + C_2\theta^{2/p}) + C_3 k\theta^2.$$

   This is principally the estimate that was given in the analysis of the h-version - cf. Introduction, eq (1.5) - with $\theta = kh$. Note also that, formally, the error is written as the sum of best approximation error plus pollution term of the order or the phase lag. However, the constant C_3 depends on p (see next section).

2. In the case of lower regularity ($l < p$) the pollution is for higher p relatively (i.e. compared to the best approximation order) still smaller as in the case of full regularity. Consider the lowest possible case of Dirac data. Then $l = 1$ and the estimate is

   $$|\tilde{e}|_1 \leq \theta(C_1 + C_2\theta^2) + C_3 k\theta^{p+1}$$

   In general, the constant C_3 depends on m. Note that, for fixed approximation order p, one is free to choose m in the range of $0, \ldots, p-1$. This can be used to optimize the size of the pollution term $C_3(p,m)(hk/2p)^{l+m+1}$.

Remark 16: We will show in the numerical evaluation that the constants C_1, C_2 are sharp. On the other hand, the theoretically predicted growth in $C_3(p)$ was not observed in several numerical examples. It is an open question whether the dual estimate is sharp in the pollution term.

We conclude this subsection with an error estimate in negative norms. We first show a mapping property; the proposition then readily follows.

Lemma 3.6 *Let* $u \in V^l$ *and* $u_{fe} \in V_h$ *be the exact and the finite element solution to the VP (1.4). Then, for* $1 \leq m \leq p-1$,

$$\|e\|_{-m} \leq DC_s(m+1)\, k \left(\frac{hk}{2p}\right)^{m+1} |e|_1 \tag{3.99}$$

where C_s *is the stability constant from Theorem 2.1 and* D *is a constant not depending on* h *and* k.

Proof : Since $X = H_o^m(\Omega)$ is a Hilbert space there exists $v_o \in X$ s.t.

$$\|e\|_{-m} = \frac{|(e, v_o)|}{|v_o|_m}.$$

Let $z \in H^{m+2}(\Omega) \cup H^1_{(o}(\Omega)$ be the solution of the VP (1.4) with data v_o. Then by Theorem 2.1. we have

$$|z|_{m+2} \leq C_s(m+1)k^m \|v_o\|_m$$

where C_s grows at most linearly with m. On $H^m_{(o}$, the full norm $\|\cdot\|_m$ is equivalent to the seminorm $|\cdot|_m$, hence there exists a constant C_1 s.t.

$$|z|_{m+2} \leq C_1 C_s(m+1)k^m |v_o|_m. \tag{3.100}$$

Further, from $B(z-\chi, e) = (v_o, e)$ for all $\chi \in V_h$ we conclude

$$\begin{aligned} \|e\|_{-m}|v_o|_m &\leq C_o(k) \inf_{\chi \in V_h} |z-\chi|_1 |e|_1 \\ &\leq Ck^2 \left(\frac{h}{2p}\right)^{m+1} |z|_{m+2}|e|_1 \end{aligned}$$

where the continuity property of the form B and the approximation property of V_h have been used. The statement now follows, inserting eq. (3.100). $\triangleleft$

Theorem 3.8 (Dual error estimate): *Let $u \in V^l$ and $u_{fe} \in V_h$ be the exact and the finite element solution to the VP (1.4). Assume $1 \leq m \leq p-1$ and $1 \leq l \leq p$.*
Then, if $hk \leq \alpha < \pi$,

$$\|e\|_{-m} \leq C(m,l)\left[C_1 + C_2 K^2 \frac{h}{2p}\right]\left(\frac{kh}{2p}\right)^{m+l+1} \|f\|_{p-1} \tag{3.101}$$

holds, where $C(m,l) = C_a(l)C_s(l)C_s(m+1)$, whereas C_1, C_2 do not depend on h, k and p.

Proof : Combining Lemma 3.7 with the preasymptotic estimate I, as given in Theorem 3.6, we obtain an estimate w.r. to $|u|_{l+1}$. The statement is then concluded from Theorem 2.1 $\triangleleft$

Remark 17: All propositions on the finite element solution contain the assumption $hk < \pi$. Essentially, we ensure herewith that the inversion of the local stiffness matrices

is well defined (hk has to be smaller than the minimal eigenvalue of the condensation). However, the inversion is also well defined if hk lies between the first and the second eigenvalue and so forth (to be more precise: is bounded away both from the first and second exact and the corresponding numerical eigenvalues). It is expected that, under this condition, *more than one halfwave* can be resolved by *one* element of higher approximation in practical computations.

4 Numerical evaluation

In this final section we present some results of numerical experiments that illustrate the theoretical results obtained in the previous section. We solve the model problem for the constant data $f \equiv 1$. The exact solution

$$u(x) = \frac{1}{k^2}\left((1-\cos kx - \sin k \sin kx) + i\,(1-\cos k)\sin kx)\right).$$

is regular and oscillating.

Best approximation in S_h^p: In Theorem 3.1 we constructed a function $s \in S_h^p(\Omega)$ that has the best H^1-approximation property to a given function $u \in H_{l+1}(\Omega)$. Writing s locally, after scaling to $I = (-1,1)$, as

$$s(\xi) = \sum_{i=1}^{p+1} c_i^{(j)} N_i(\xi).$$

we have, by interpolating property, $c_1^{(j)} = u_{j-1}$ and $c_2^{(j)} = u_j$. Further, by orthogonality, $c_i^{(j)} = (u'^{(j)}, N_i')$. and $|e_{ba}|_1^2 = |u-s|_1^2 = |u|_1^2 - |s|_1^2$ with

$$|s|_1^2 = \sum_{j=1}^{n}\left(h|D^j u|^2 + \frac{2}{h}\sum_{i=3}^{p_j+1}|c_i^{(j)}|^2\right).$$

The approximation property, as given in Theorem 3.1, reads for $m=1$ and $l=p$ as

$$|\tilde{e}|_1 := \frac{|u-s|_1}{|u|_1} \;\leq\; C_a(p)\left(\frac{h}{2p}\right)^p \frac{|u|_{p+1}}{|u|_1} \tag{4.1}$$

where C_a is the approximation constant, growing as $(e/2)^p/p^(1/4)$.

In Fig. 1 the error $|\tilde{e}|_1$ is plotted for $k = 50$ and $hk < \pi$. The slopes are $-p$, in accordance with eq (4.1).

Consider next a plot for the approximation error on coarse grid ($hk > \pi$ - Fig. 2). We observe an interesting effect whenever the mesh locally 'hits' the size of a halfwave (two, three, ...) halfwaves. To explain this effect, note that for $k = n\pi$ (m-integer) the solution u reduces to

$$u = \frac{1}{k^2}(1 - \cos kx).$$

Hence in this case we locally solve the problem of best H^1-approximation of $u_{loc} = \cos x$ on one, two, three, ... halfwaves. For even n, this approximation problem is symmetric and has identical solutions for even/odd p (lines I, III in Fig. 2). For odd n, the problem is antimetric and has identical solutions for odd/even p (line II in Fig.2). We have choosen $k = 2m\pi$ to highlight the character of the observation. The same effect occurs, however, for any k if either k or kh are close to integer multitudes of π.

Returning to the estimate in Theorem 3.1. - here: eq (4.1) - we next present numerical data illuminating the dependence of the constant C_a on p. Namely, for $p = 1 \ldots 6$ and $hk = 1$ (case $n = k = 50$) we record the relative error, the measured value of $C_{1,meas}(p) = |\tilde{e}|_{1,meas}(2p)^{2p}$ and the predicted $C_1(p)$ from eq (3.5).

Table 1: Constant $C_a(p)$ in the Approximation Theorem: magnitude as computed from measured data compared to magnitude as computed from theoretical prediction in eq (3.5)

p	$\|\tilde{e}\|_{1,meas}$	$C_{a,meas}$	C_a, (3.5)
1	0.2823	0.5646	1.02
2	0.367E-1	0.5872	1.17
3	0.3095E-2	0.6685	1.43
4	0.1965E-3	0.9049	1.81
5	0.9829E-5	0.9829	2.33
6	0.4135E-6	1.2357	3.02

We see that, for the particular case computed, both the magnitudes of the measured constant and it's growth rates with p are lower than the upper theoretical estimates. We conclude that for the example under consideration, no further growth with p occurs in the relative error due to the ratio $|u|_{p+1}/|u|_1$.

For graphical illustration we compare in Fig. 3 (for $k = 30\pi$ and $m = 50$) the estimated and measured errors for $p = 1\ldots6$. Namely, we plot (setting $C_a(p) \equiv 1$)

$$\mathrm{est}_n(p) = (n \cdot 2p)^{-p}$$

for $n = 50$ and $1 \leq p \leq 6$. We compare with the relative error $|\tilde{e}|_1$ as obtained from computation. The measured error is in close agreement with the estimator, indicating that the estimate (4.1) is sharp.

Finally, we relate the error of best approximation to the number of degrees of freedom of the discrete model. In one dimension, this number is $d := h^{-1}p$. For fixed k, the error estimate (4.1) becomes

$$|\tilde{e}|_1 \leq \frac{C(p)}{2^p} d^{-p}. \tag{4.2}$$

Observe in Fig. 4 both the predicted rate of convergence w.r. to d and the decrease of the multiplicating factor with the increase of p.

Error of the finite element solution: The error of the finite element solution is computed by

$$\begin{aligned} |e_{fe}|_1^2 &= |u - u_{fe}|_1^2 \\ &= e_1^2 - \sum_{j=1}^{n} \frac{2}{\Delta_j} \sum_{i=3}^{p_j+1} \left(\bar{a}_i^{(j)} c_i^{(j)} + a_i^{(j)} \bar{c}_i^{(j)} - \bar{a}_i^{(j)} a_i^{(j)} \right) \end{aligned} \tag{4.3}$$

where

$$e_1^2 = |u|_1^2 - h \sum_{j=1}^{n} (D_j u D_j \bar{u}_h + D_j u_h D_j \bar{u} - D_j u_h D_j \bar{u}_h) \tag{4.4}$$

is the error of piecewise linear approximation (cf. part I) and $a_i^{(j)}$ are the coefficients of the bubble modes in the local finite element ansatz, cf. subsection 3.2.

In Fig. 5, the relative error of the finite element solution is plotted against the relative error of the best approximation. The wavenumber is $k = 30\pi$ and the results are compared for $p = 1\ldots6$. We clearly see the optimal convergence of the finite element solution for sufficiently small h. In the given example, the optimality constant C in Theorem 3.5 is asymptotically 1 as the figure shows.

To illustrate the behaviour in the preasymptotic range, consider the horizontal line drawn at $\tilde{e} \equiv 0.1$. Compared to the asymptotic behavior with the optimality constant close to 1, the finite element solution is significantly polluted on the preasymptotic error level. We give the numerical results in Table 2.

The error estimate for oscillating solutions has been given in Corollary 3.2, eq (3.98). For our solution, the special case 1. applies, hence we have the estimate

$$|\tilde{e}|_1 \leq C_1(p)\theta + C_2(p)k\theta^2 \ ^5$$

with

$$\theta = \left(\frac{kh}{2p}\right)^p .$$

In this inequality, $C_1(p)\theta$ estimates the error of best approximation as discussed in the previous paragraph. The second member $C_2(p)k\theta^2$ reflects the pollution and is of the same order as the phase lag (cf. Theorem 3.2). Theoretically, the constant C_2 may significantly grow with p, cf. Theorem 3.7 vs. Lemma 3.5. As the table shows, we do not observe this growth in the example considered. As commented before, we cannot exclude that the theoretically established dependence of $C_2(p)$ on p is due to a technicality in the proof.

Finally we observe that the number of elements for which the finite element error is a fixed magnitude (given tolerance) decreases significantly with increasing p.

Table 2: Errors of finite element solution and best approximation at $\tilde{e}_{fe} \approx 0.1$ for $p = 1 \dots 6$.

[5] We neglect the term $C_1(p)\theta^{1+2/p}$ of eq (3.98)

p	1	2	3	4	5	6
$\tilde{e}_{fe}^{(meas)}$	0.099947	0.09956	0.09683	0.0911	0.0829	0.09833
n (# of elem.)	491	76	35	22	16	12
# of DOF	491	152	105	88	80	72
$\theta = \left(\frac{k}{2np}\right)^p$	0.09597	0.09612	0.0904	0.0822	0.07092	0.7861
$\tilde{e}_{ba}^{(meas)}$	0.05538	0.05607	0.05640	0.05543	0.0520	0.0626
$\tilde{e}_{fe} - \tilde{e}_{ba}$	0.044567	0.04349	0.04040	0.03567	0.02380	0.0380
$k\theta^2$	0.8681	0.8707	0.7702	0.6373	0.474	0.5823
$C(p) = \frac{\tilde{e}_{fe}-\tilde{e}_{ba}}{k\theta^2}$	0.0513	0.050	0.0525	0.056	0.0502	0.0656

Acknowledgement: This work was supported by ONR Grant N00014-93-I-0131. The first author was supported partly by Grant No 517 402 524 3 of the German Academic Exchange Service (DAAD). The support is kindly acknowledged.

References

[AKS] A.K. Aziz, R.B. Kellogg and A.B. Stephens, A two point boundary value problem with a rapidly oscillating solution, Numer. Math. 53, 107-121 (1988)

[BayGTu] Bayliss, A.; Goldstein, C.I.; Turkel, E., On accuracy conditions for the numerical computation of waves, J. Comp. Phys. 59 (1985), 396-404

[BA] I. Babuška and A.K. Aziz, The mathematical foundations of the finite element method, in: A.K. Aziz (ed.), The mathematical foundations of the finite element method with applications to partial differential equations, Academic Press, New York 1972, pp. 5-359

[BKS] I. Babuška, I.N. Katz and and B.S. Szabó, Finite element analysis in one dimension, Lecture Notes, to appear

[Be] R. Bellman, A note on an inequality of E. Schmidt, Bulletin of the American Mathematical Society, Vol. 50, No. 10 (October 1944), 734-736

[Bu] D. Burnett, A 3-D acoustic infinite element based on a generalized multipole expansion, Journ. Acoust. Soc.

[De] Demkowicz, L.,Asymptotic convergence in finite and boundary element methods: part I: theoretical results, preprint 1993

[DSSS] J. Douglas Jr., J.E. Santos, D. Sheen, L. Schreiyer, Frequency domain treatment of one-dimensional scalar waves, Mathematical Models and Methods in Applied Sciences, Vol. 3, No. 2 (1993) 171-194

[G] D. Givoli, Numerical methods for problems in infinite domains, Elsevier Publ. Amsterdam 1992

[HH] I. Harari and T.J.R. Hughes, Finite element method for the Helmholtz equation in an exterior domain: model problems, Comp. Meth. Appl. Mech. Eng. 87 (1991), 59-96

[IB] F. Ihlenburg and I. Babuška, Finite element solution to the Helmholtz equation with high wavenumber - part I: The h-version of the FEM, Technical Note BN-1159 (1993), Institute for Physical Science and Technology, University of Maryland at College Park

[N] I.P. Natanson, Konstruktive Funktionentheorie, Akademieverlag Berlin 1955

[Sch] A.H. Schatz, An analysis of the finite element method for second order elliptic boundary value problems, in: A.H. Schatz, V.T. Thomée and W. L. Wendland, Mathematical theory of finite and boundary element methods, Birkhäuser Verlag 1990

[SB] B. Szabó and I. Babuška, Finite Element Analysis, J. Wiley, New York etc., 1991

[TP1] L.L. Thompson and P.M. Pinsky, Complex wavenumber Fourier analysis of the p-version finite element method, Computational Mechanics, 13 (1994), 255 -275

[TP2] L.L. Thompson and P.M. Pinsky, A Galerkin Least Squares Finite Element Method for the Two-Dimensional Helmholtz Equation, Int. J. Num. Meth. Engng. (to appear)

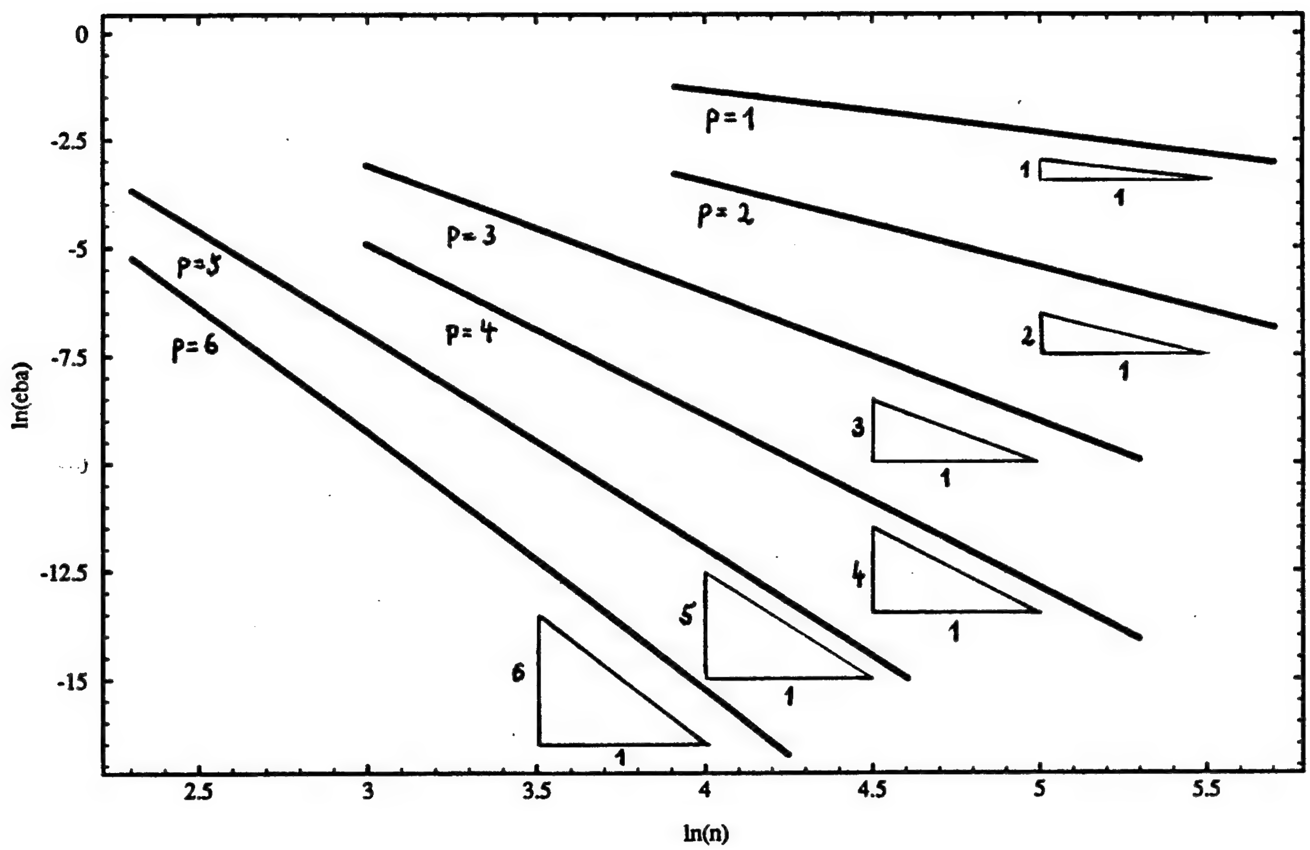

Figure 1: Relative error (eba) of the best H^1-approximation in $S_h^p(\Omega)$ to the exact solution. Rates of convergence in H^1-seminorm for $p = 1, 2, \ldots, 6$ ($k = 50$, $n = h^{-1}$ - number of elements).

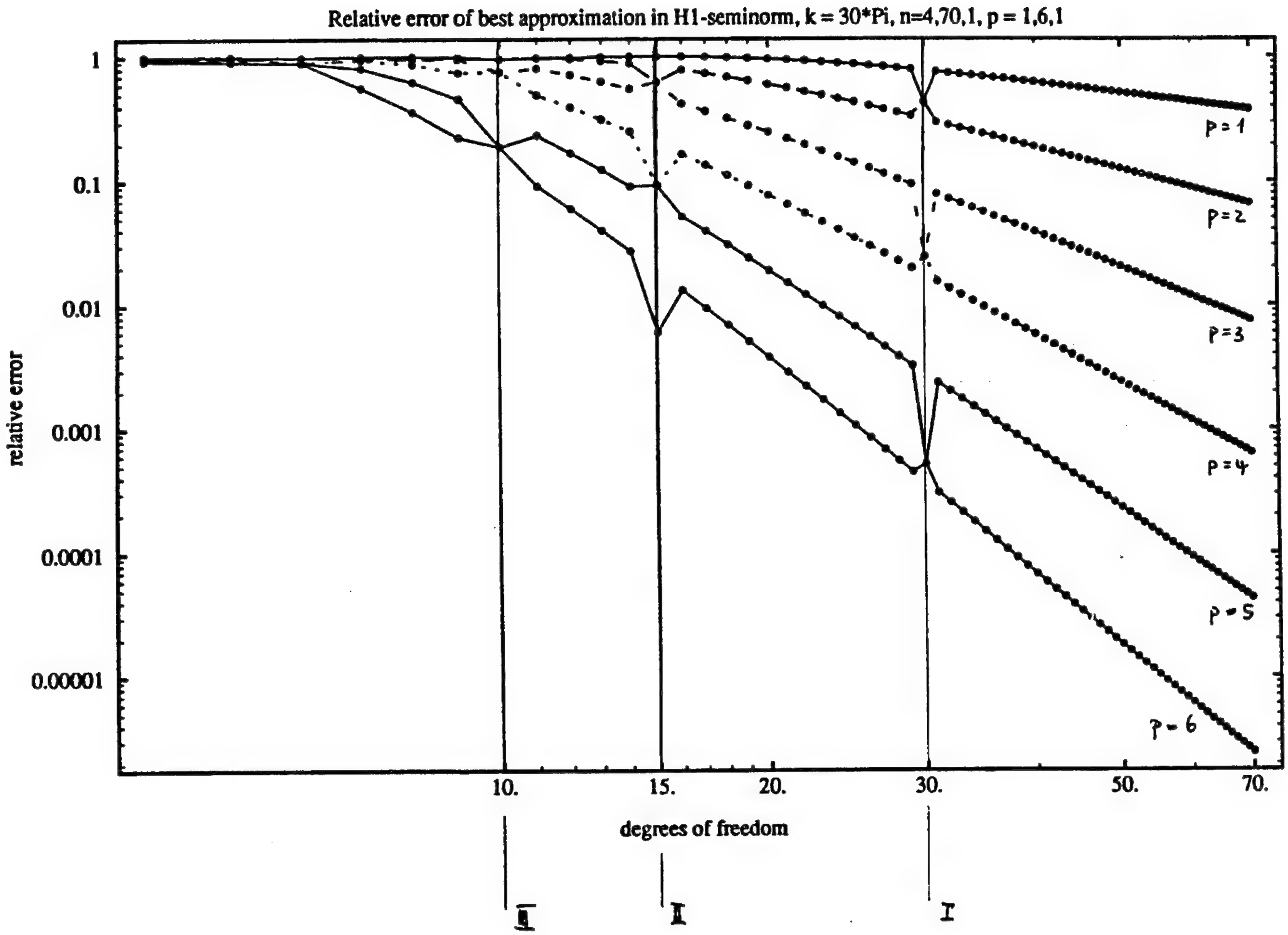

Figure 2: Relative error of the best approximation in H^1-seminorm on coarse grid for $k = 30\pi$.

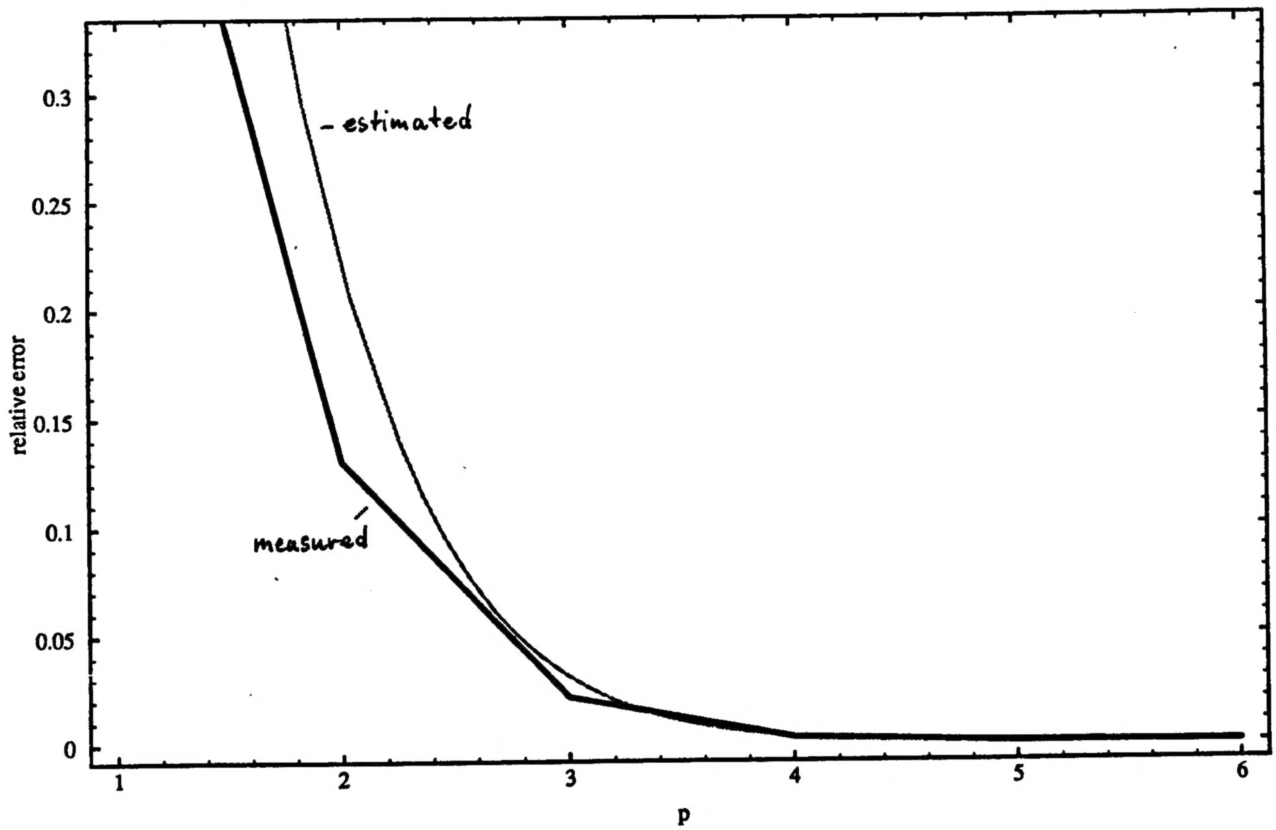

Figure 3: Relative error of the best approximation in H^1-seminorm: estimated vs. measured values for $k = 30\pi$, $n = 50$ and $p = 1, 2, \ldots, 6$.

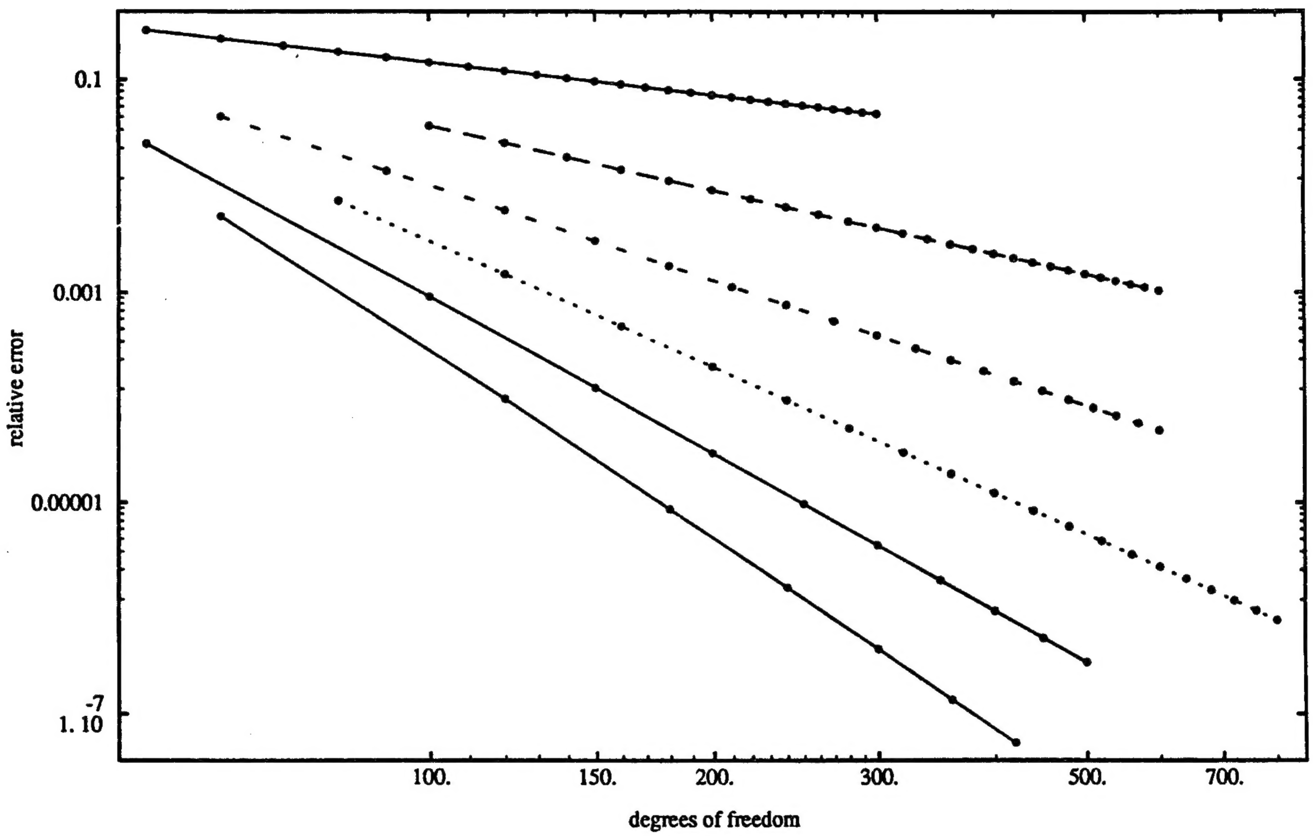

Figure 4: Relative error of best approximation vs. number of degrees of freedom for $k = 50$ and $p = 1, 2, \ldots, 6$.

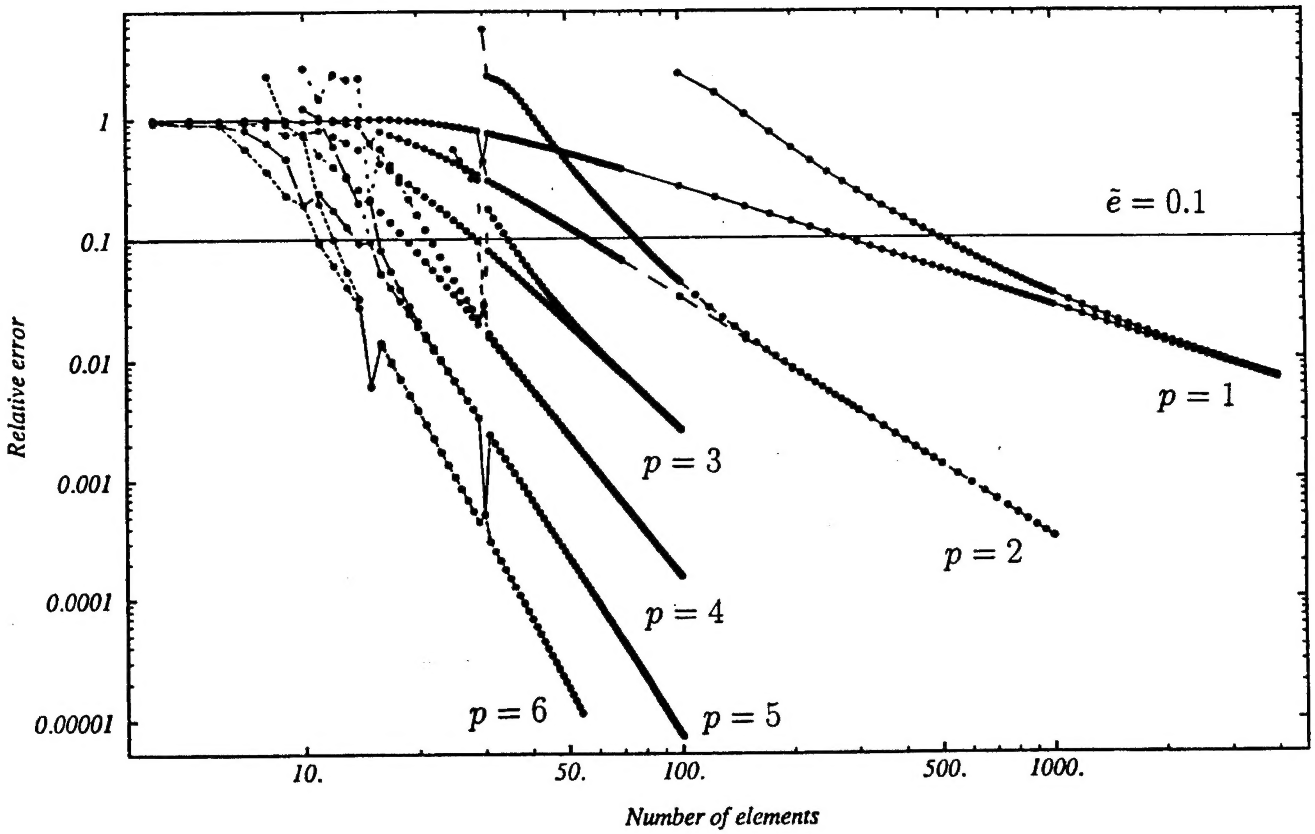

Figure 5: Relative error of the finite element solution versus best approximation error for $k = 30\pi$ and $p = 1, 2, \ldots, 6$.

The Laboratory for Numerical Analysis is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

> To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.
>
> To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.
>
> To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.
>
> To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Institute of Standards and Technology.
>
> To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.).

Further information may be obtained from **Professor I. Babuška**, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742-2431.